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A Direct Proof of Inada-Sen-Pattanaik Theorem on Majority Rule

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Abstract

A restriction on preferences called Latin Square partial agreement is introduced. It is shown that the Latin Square partial agreement (i) is logically equivalent to the union of value-restriction, limited agreement and extremal restriction (ii) is necessary and sufficient for quasi-transitivity of the social preference relation generated by the majority rule and if the number of concerned individuals is odd for every triple of alternatives then it is necessary and sufficient for transitivity.

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The purpose of this paper is to provide a direct and unified proof for the Inada-Sen-Pattanaik theorem establishing necessary and sufficient conditions for quasi-transitivity and transitivity of the social preference relation generated by the majority rule. The theorem states that value-restriction, limited agreement and extremal restriction constitute a set of necessary and sufficient conditions for the quasi-transitivity of the social preference relation yielded by the majority rule, and if the number of concerned voters is odd for every triple of alternatives then the same three restrictions constitute a set of necessary and sufficient conditions for transitivity. We introduce a restriction on preferences called Latin Square partial agreement (LSPA), which is logically equivalent to the union of value-restriction, limited agreement and extremal restriction, and prove the theorem directly in terms of this condition. There is considerable gain both in terms of simplicity and clarity. Instead of separate proofs for the sufficiency of value-restriction, limited agreement and extremal restriction, a unified proof is provided. Simplification of proof of the necessity part is even greater as it is no longer necessary to systematically consider all violations of the three restrictions for arriving at the set of all configurations which violate all three restrictions. Relating quasi-transitivity and transitivity to a single condition on preferences provides added insight into the structure of the majority rule.

1. Restrictions on Preferences:

The set of social alternatives would be denoted by S. The cardinality n of S would be assumed to be finite and greater than 2. The set of individuals and the number of individuals are designated by L and N respectively. N is assumed to be finite and greater than 2. N ( ) would stand for the number of individuals holding the preferences specified in the parenthesis and N_k for the number of individuals holding the k-th preference ordering. Each individual $i \in L$ is assumed to have an ordering R_i defined over S. The symmetric and asymmetric parts of R_i are denoted by I_i and P_i respectively. The social preference relation is denoted by R and its symmetric and asymmetric components by I and P respectively.

Majority Rule : $\forall\, x, y \in S$: $x\ R\ y$ iff $N(xP_iy) \geqslant N(yP_ix)$.

An individual is defined to be concerned with respect to a triple iff he is not indifferent over every pair of alternatives belonging to the triple, otherwise he is unconcerned.

Value-Restriction (VR) : VR holds over $\{x, y, z\}$ iff $\exists$ distinct a, b, c $\in \{x, y, z\}$ such that $[(\forall$ concerned i : $b P_i a \vee c P_i a) \vee (\forall$ concerned i : $(a P_i b \wedge a P_i c) \vee (b P_i a \wedge c P_i a)) \vee (\forall$ concerned i : $a P_i b \vee a P_i c)]$.

Limited Agreement (LA) : LA holds over $\{x, y, z\}$ iff $\exists$ distinct a, b, $\in \{x, y, z\}$ such that $\forall i : a R_i b$.

Extremal Restriction (ER) : ER holds over $\{x, y, z\}$ iff $\forall$ a, b, c $\in \{x, y, z\}$: $[(\exists i : a P_i b P_i c) \rightarrow \forall i : (c P_i a \rightarrow c P_i b P_i a)]$.

There are 13 logically possible orderings of a triple $\{x, y, z\}$, listed below. Orderings 1 to 12 are concerned.

1. $x P_i y P_i z$
2. $x P_i z P_i y$
3. $y P_i x P_i z$
4. $y P_i z P_i x$
5. $z P_i x P_i y$
6. $z P_i y P_i x$
7. $x P_i y I_i z$
8. $y I_i z P_i x$
9. $y P_i x I_i z$
10. $x I_i z P_i y$
11. $z P_i x I_i y$
12. $x I_i y P_i z$
13. $x I_i y I_i z$

Following are the six logically possible R-orderings of $\{x, y, z\}$:

(i) $x R_i y R_i z$

(ii) $y R_i z R_i x$

(iii) $z R_i x R_i y$

(iv) $x R_i z R_i y$

(v) $z R_i y R_i x$

(vi) $y R_i x R_i z$

(i), (ii) and (iii) form the first Latin Square LS (xyzx) and (iv), (v) and (vi) the second Latin Square LS (xzyx). We will denote the set of concerned orderings of Latin Square (xyzx) by T (xyzx) and the set of concerned orderings of Latin Square (xzyx) by T(xzyx). From the set of concerned orderings (1), (4), (5), (7), (8), (9), (10), (11) and(12) belong to T (xyzx) while (2), (3), (6), (7), (8), (9), 10), (11), and (12) belong to T (xzyx).

A set W of orderings defined over a triple $\{x,y,z\}$ corresponds to the configuration of individual orderings $\{R_1 \ldots R_N\}$ over $\{x,y,z\}$ iff every ordering in W is held by some $i \in L$ and if an ordering is held by some $i \in L$ then it belongs to W. $\overline{W}$ would denote the set of concerned orderings in W.

Latin Square Partial Agreement (LSPA) : If the set of concerned individual orderings over $\{x,y,z\}$ contains a Latin Square involving a strong ordering then there exist distinct a, b $\in \{x,y,z\}$ such that any individual who holds an ordering belonging to the Latin Square in question considers a to be at least as good as b. Formally, LSPA holds over $\{x,y,z\}$ iff $\forall$ a,b,c $\in \{x,y,z\}$:

$[(\exists$ concerned $i,j,k \in L$: $a P_i b P_i c \wedge b R_j c R_j a \wedge c R_k a R_k b) \rightarrow (\forall R_i \in \overline{W} \cap T\,(abca): a R_i c)]$.

Theorem 1 : LSPA is logically equivalent to the union of VR, LA, and ER.

Proof: First we show that LSPA $\rightarrow$ (VR $\vee$ LA $\vee$ ER). Let LSPA be satisfied. If the set of concerned R_i does not contain a Latin Square then VR is satisfied as VR is violated iff the set of concerned R_i contains a Latin Square.

Now suppose that the set of concerned R_i contains exactly one Latin Square, say, LS (xyzx). Then it follows that $[\exists$ concerned $i,j,k : x R_i z R_i y \wedge z R_j y R_j x \wedge y R_k x R_k z]$ must be false. Without any loss of generality assume that there does not exist any concerned individual who holds $x R_i z R_i y$.

$$\sim(\exists \text{concerned } i: x R_i z R_i y) \tag{1}$$

$$\rightarrow [\forall \text{ concerned } i : [(xR_iz \rightarrow yP_iz) \wedge (zR_iy \rightarrow zP_ix)]]$$

$$\rightarrow \forall \text{ concerned } i : [(xR_iyR_iz \rightarrow xR_iyP_iz) \wedge (zR_ixR_iy \rightarrow zP_ixR_iy)]. \tag{2}$$

If no $R_i \in \overline{W} \cap T$ (xyzx) is strong then (1) and (2) imply that $\forall i \in L : y R_i x$ and therefore LA is satisfied. Next suppose that some $R_i \in \overline{W} \cap T$ (xyzx) is strong. Then, as LSPA is satisfied there must exist distinct $a, b \in \{x, y, z\}$ such that $\forall R_i \in \overline{W} \cap T$ (xyzx) : $a R_i b$. Given the fact that there are concerned individuals who hold $x\ R_i\ y\ P_i\ z$, $y\ R_i\ z\ R_i\ x$, and $z\ P_i\ x\ R_i\ y$, it follows that $(a,b) = (y,x)$. As no concerned individual holds $x\ R_i\ z\ R_i\ y$ it follows that for every R_i not belonging to $\overline{W} \cap T\ (xyzx)$, also, $y\ R_i\ x$ holds. Thus $\forall i \in L$: $y\ R_i\ x$ which implies that LA holds.

Finally consider the case when both the Latin Squares are contained in the set of concerned R_i. If no individual has a strong ordering then ER is satisfied. Next suppose that someone holds a strong ordering, say, $x P_i\ y\ P_i\ z$.

$$\exists i\ :\ x\ P_i\ y\ P_i\ z \qquad (3)$$

As LSPA is satisfied there must exist distinct $a,b \in \{x,y,z\}$ such that $\forall\ R_i \in \overline{W} \cap T\ (xyzx)$: $a\ R_i\ b$. In view of the fact that $\exists\ i : x\ P_i\ y\ P_i\ z$, it follows that $(a,b) = (x,z)$. Therefore,

$$\forall i\ :\ (\ yR_izR_ix \longrightarrow yR_izI_ix) \qquad (4)$$

and $$\forall i\ :\ (\ zR_ixR_iy \longrightarrow zI_ixR_iy\ ) \qquad (5)$$

As the set of concerned R_i contains both the Latin Squares it follows that there exists a concerned individual for whom $z\ R_i\ y\ R_i\ x$ holds.

(4) and (5) imply that $\forall$ concerned i : $(zR_iyR_ix \longrightarrow zP_iyP_ix)$. Thus,

$$\exists i\ :\ z\ P_i\ y\ P_i\ x \qquad (6)$$

By LSPA then there exist distinct $a,b \in \{x,y,z\}$ such that $\forall R_i \in \overline{W} \cap T\ (xzyx)$: $a\ R_i\ b$. As $\exists i : z\ P_i\ y\ P_i\ x$, $(a,b) = (z,x)$. Therefore,

$$\forall i \ : \quad (x R_i z R_i y \longrightarrow x I_i\ z R_i y\ ) \tag{7}$$

and

$$\forall i \ : \quad (y R_i x R_i z \longrightarrow y R_i x I_i z\ ) \tag{8}$$

From (3) through (8) we conclude that the set of concerned R_i is $\{ xP_iyP_iz,\ \ zP_iyP_ix,\ \ yP_ixI_iz,\ \ xI_izP_iy \}$. This configuration satisfies ER.

Thus whenever LSPA is satisfied at least one of the restrictions VR, LA and ER holds.

Next we prove that $(VR \vee LA \vee ER) \rightarrow$ LSPA. Now LSPA is violated iff the set of concerned R_i contains a Latin Square involving a strong ordering and for all distinct $a,b \in \{x,y,z\}$ there exist R_i, R_j belonging to the Latin Square in question such that $a\ P_i\ b$ and $b\ P_j\ a$. That is to say, LSPA is violated iff the set of concerned R_i contains one of the following six 3-ordering sets, except for a formal interchange of alternatives.

(A)		(B)		(C)	
	$x\ P_i\ y\ P_i\ z$		$x\ P_i\ y\ P_i z$		$x\ P_i\ y\ P_i\ z$
	$y\ P_i\ z\ P_i\ x$		$y\ P_i\ z\ P_i\ x$		$y\ P_i\ z\ P_i\ x$
	$z\ P_i\ x\ P_i\ y$		$z\ P_i\ x\ I_i\ y$		$z\ I_i\ x\ P_i\ y$

(D) $x P_i y P_i z$
$y P_i z I_i x$
$z P_i x I_i y$

(E) $x P_i y P_i z$
$y I_i z P_i x$
$z P_i x I_i y$

(F) $x P_i y P_i z$
$y I_i z P_i x$
$z I_i x P_i y$

As each one of these sets violates all 3 restrictions VR, LA, and ER, it follows that $(VR \vee LA \vee ER) \rightarrow LSPA$.

2. Necessary and Sufficient Condition for Quasi-Transitivity

Theorem 2 : A necessary and sufficient condition for quasi-transitivity of the social preference relation generated by the majority rule is that the Latin Square Partial Agreement holds over every triple of alternatives.

Proof : Suppose quasi-transitivity is violated. Then there exist x,y,z such that $xPy \wedge yPz \wedge \sim(xPz)$. Let Nc denote the number of individuals who are concerned with respect to the triple $\{x,y,z\}$.

$$x P y \rightarrow N(xP_iy) > N(yP_ix) \quad (1)$$

$$\rightarrow N(\text{concerned } i \text{ with } xR_iy) > N(\text{concerned } i \text{ with } yR_ix)$$

$$\rightarrow N(\text{concerned } i \text{ with } xR_iy) > Nc/2 \quad (2)$$

$$y P z \rightarrow N(yP_iz) > N(zP_iy) \quad (3)$$

$$\rightarrow N(\text{concerned } i \text{ with } yR_iz) > Nc/2 \quad (4)$$

$\sim(x\ P\ z) \rightarrow N\ (zP_i x)\ \geq\ N\ (xP_i z)$ (5)

$\rightarrow N$ (concerned i with $zR_i x$)

$\geq\ Nc/2$ (6)

(2) and (4) $\rightarrow \exists$ concerned i : $x\ R_i\ y\ R_i\ z$ (7)

(4) and (6) $\rightarrow \exists$ concerned i : $y\ R_i\ z\ R_i\ x$ (8)

(2) and (6) $\rightarrow \exists$ concerned i : $z\ R_i\ x\ R_i\ y$ (9)

(7), (8), and (9) imply that $\overline{W}$ contains the Latin Square (xyzx) (10)

(1), (2), and (3) imply

$$N_1 + N_2 + N_5 + N_7 + N_{10} > N_3 + N_4 + N_6 + N_8 + N_9 \quad (11)$$

$$N_1 + N_3 + N_4 + N_9 + N_{12} > N_2 + N_5 + N_6 + N_{10} + N_{11} \quad (12)$$

$$N_4 + N_5 + N_6 + N_8 + N_{11} \geq N_1 + N_2 + N_3 + N_7 + N_{12} \quad (13)$$

By adding inequalities (11), (12), and (13), we obtain

$$N_1 + N_4 + N_5 > N_2 + N_3 + N_6 \quad (14)$$

Therefore, at least one of the orderings belonging to $\overline{W} \cap T$ (xyzx) is a strong ordering. (15)

Adding (11) and (12), (12) and (13), and (11) and (13), we obtain

$$2N_1 + N_7 + N_{12} > 2N_6 + N_8 + N_{11} \quad (16)$$

$$2N_4 + N_8 + N_9 > 2N_2 + N_7 + N_{10} \quad (17)$$

$$2N_5 + N_{10} + N_{11} > 2N_3 + N_9 + N_{12} \quad (18)$$

Adding (16) and (17), (17) and (18), and (16) and (18), we get

$$2\,N_1 + 2\,N_4 + N_9 + N_{12} > 2\,N_2 + 2\,N_6 + N_{10} + N_{11} \quad (19)$$

$$2\,N_4 + 2\,N_5 + N_8 + N_{11} > 2\,N_2 + 2\,N_3 + N_7 + N_{12} \quad (20)$$

$$2\,N_1 + 2\,N_5 + N_7 + N_{10} > 2\,N_3 + 2\,N_6 + N_8 + N_9 \quad (21)$$

Now,

$$(7) \longrightarrow \exists\, R_i \in \overline{W} \cap T\,(xyzx) : x\,P_i\,z \quad (22)$$

$$(8) \longrightarrow \exists\, R_i \in \overline{W} \cap T\,(xyzx) : y\,P_i\,x \quad (23)$$

$$(9) \longrightarrow \exists\, R_i \in \overline{W} \cap T\,(xyzx) : z\,P_i\,y \quad (24)$$

$$(19) \longrightarrow 2\,N_1 + 2\,N_4 + N_9 + N_{12} > 0$$

$$\longrightarrow \exists\, R_i \in \overline{W} \cap T\,(xyzx) : y\,P_i\,z \quad (25)$$

$$(20) \longrightarrow 2\,N_4 + 2\,N_5 + N_8 + N_{11} > 0$$

$$\longrightarrow \exists\, R_i \in \overline{W} \cap T\,(xyzx) : z\,P_i\,x \quad (26)$$

$$(21) \longrightarrow 2\,N_1 + 2\,N_5 + N_7 + N_{10} > 0$$

$$\longrightarrow \exists\, R_i \in \overline{W} \cap T\,(xyzx) : x\,P_i\,y \quad (27)$$

(22) through (27) imply that there do not exist distinct $a, b \in \{x, y, z\}$ such that $\forall\, R_i \in \overline{W} \cap T\,(xyzx) : a\,R_i\,b$. (28)

From (10), (15) and (28) we conclude that LSPA is violated. Thus it has been shown that violation of quasi-transitivity implies violation of LSPA, i.e. LSPA is a sufficient condition for quasi-transitivity.

From the definition of LSPA it follows that it is violated iff the set of R_i contains one of six sets A, B, C, D, E, and F mentioned in the proof of Theorem 1. To prove the necessity of LSPA, therefore, it suffices to show that for each of these sets there exists an assignment of individuals which results in violation of quasi-transitivity.

For (A) take $N_1 + N_3 > N_2$, $N_1 + N_2 > N_3$, $N_2 + N_3 > N_1$, for (B) $N_1 > N_2$, $N_1 + N_2 > N_3$, $N_2 + N_3 > N_1$, for (C) $N_1 + N_3 > N_2$, $N_1 + N_2 > N_3$, $N_2 > N_1$, for (D) $N_3 > N_1 > N_2$, $N_1 + N_2 > N_3$, for (E) $N_1 > N_2$, $N_1 > N_3$, $N_2 + N_3 > N_1$, and for (F) $N_2 > N_1 > N_3$, $N_1 + N_3 > N_2$; where subscript $i = 1, 2, 3$ refers to the i th ordering in the set in question. Then the social preference relation generated by the majority rule is $xPy \wedge yPz \wedge zPx$ which violates quasi-transitivity.

3. Necessary and sufficient Condition for Transitivity

Theorem 3 : If for every triple the number of concerned individuals is odd then a necessary and sufficient condition for transitivity of the social preference relation generated by the majority rule is that over every triple Latin Square Partial Agreement holds.

Proof : Suppose transitivity is violated. Then for some x,y,z we must have

$$xRy \wedge yRz \wedge zPx$$

$xRy \rightarrow N\ (x\ P_i\ y) \geqslant N\ (y\ P_i\ x)$

$\rightarrow N$ (concerned i with $x\ R_i\ y$) $\geqslant$

N (concerned i with $y\ R_i x$)

$\rightarrow N$ (concerned i with $x\ R_i\ y$) $\geqslant$ $Nc/_2$,

where Nc = number of individuals concerned with respect to $\{x,y,z\}$

$\rightarrow N$ (concerned i with $x\ R_i\ y$) $>$ $Nc/_2$, (1)

as Nc is odd.

Similarly, $yRz \rightarrow N$ (concerned i with $y\ R_i z$) $>$ $Nc/_2$ (2)

$zPx \rightarrow N$ (concerned i with $zR_i x$) $>$ $Nc/_2$ (3)

(1) and (2) $\rightarrow$ $\exists$ concerned i : $x\ R_i\ y\ R_i\ z$ (4)

(2) and (3) $\rightarrow$ $\exists$ concerned i : $y\ R_i\ z\ R_i\ x$ (5)

(1) and (3) $\rightarrow$ $\exists$ concerned i : $z\ R_i\ x\ R_i\ y$ (6)

$xRy \rightarrow N_1 + N_2 + N_5 + N_7 + N_{10} \geqslant N_3 + N_4 + N_6 + N_8 + N_9$ (7)

$yRz \rightarrow N_1 + N_3 + N_4 + N_9 + N_{12} \geqslant N_2 + N_5 + N_6 + N_{10} + N_{11}$ (8)

$zPx \rightarrow N_4 + N_5 + N_6 + N_8 + N_{11} > N_1 + N_2 + N_3 + N_7 + N_{12}$ (9)

(7), (8) and (9) $\rightarrow$ $N_1 + N_4 + N_5 > 0$ (10)

(7) and (8) $\rightarrow 2\ N_1 + N_7 + N_{12} \geqslant 2\ N_6 + N_8 + N_{11}$ (11)

(8) and (9) $\rightarrow 2\ N_4 + N_9 + N_8 > 2\ N_2 + N_{10} + N_7$ (12)

(7) and (9) $\rightarrow 2\ N_5 + N_{10} + N_{11} > 2\ N_3 + N_9 + N_{12}$ (13)

Now,

(11) and (12) $\rightarrow N_1 + N_4 + N_9 + N_{12} > 0 \rightarrow \exists R_i \in$

$\overline{W} \cap T$ (xyzx) : $y P_i z$ (14)

(12) and (13) $\rightarrow N_4 + N_5 + N_8 + N_{11} > 0 \rightarrow \exists R_i \in$

$\overline{W} \cap T$ (xyzx) : $z P_i x$ (15)

(11) and (13) $\rightarrow N_1 + N_5 + N_7 + N_{10} > 0 \rightarrow \exists R_i \in$

$\overline{W} \cap T$ (xyzx) : $x P_i y$ (16)

(4), (5), (6), and (10) imply that the set of R_i contains the Latin Square (xyzx) involving a strong ordering. (4), (5), (6) alongwith (14), (15), (16) imply that there do not exist distinct a, b $\in \{x,y,z\}$ such that $\forall R_i \in \overline{W} \cap T$ (xyzx) : $a R_i b$. Thus LSPA is violated, which establishes the sufficiency part. Proof of the theorem is completed by noting that the assignments of the necessity part of the theorem 2 are consistent with there being an odd number of concerned individuals.