The Coherence of Rights

Satish K. Jain
Centre for Economic Studies & Planning
Jawaharlal Nehru University
New Delhi 110067
INDIA

Abstract

A rights-assignment is called coherent iff the exercise of rights by itself never leads to an empty choice set irrespective of which profile of individual orderings and which nonempty finite subset of alternatives are considered. We discuss the following two formalizations of the idea of coherence : (i) a rights-assignment $D = (D_1,...,D_n)$ is coherent iff for every profile of individual orderings $(R_1,...,R_n)$, there exists an ordering-extension of each and every $D_i \cap R_i$, where D_i is the set of ordered pairs assigned to individual i and R_i is i's ordering of social alternatives (ii) a rights-assignment D is coherent iff there is no critical loop in D. We show that neither of the two formalizations is equivalent to coherence. We present modified versions of these formalizations and show them to be equivalent to coherence.

We discuss some of the implications of our analysis for the way the idea of a liberal individual is formalized. We introduce a new formalization of the idea of a liberal individual. Using it, we show the existence of a collective choice rule satisfying (i) unrestricted domain (ii) conditional weak Pareto-criterion (iii) coherent libertarianism and (iv) the property that, whenever profile of individual orderings is such that the weak Pareto-criterion and coherent libertarianism do not conflict, the choice set is a subset of Pareto-optimal alternatives in the sense of weak Pareto-criterion.

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Introduction

The main concern of this paper is with the idea of coherence of rights-assignments and its formalizations. A rights-assignment is called coherent if and only if the exercise of rights by itself never leads to an empty choice set no matter which profile of individual orderings and which nonempty finite subset of alternatives are considered. In other words a rights-assignment is coherent if and only if Gibbard paradox [3] is not possible for any profile of individual orderings and for any nonempty finite subset of alternatives.

We discuss two of the formalizations in the social choice literature of the idea of coherence of rights-assignments and show that neither of the two formalizations is equivalent to coherence of rights-assignments, if by coherence of rights-assignments we mean the absence of Gibbard paradox for all profiles of individual orderings and for all nonempty finite subsets of alternatives. Let D = $(D_1,...,D_n)$ be a rights-assignment, where D_i is the set of ordered pairs assigned to individual i (individual i's protected sphere), and $(R_1,...,R_n)$ a profile of individual orderings. One formalization states that D is coherent if and only if there exists an ordering-extension of each and every $D_i \cap R_i$.[1] We show that while the existence of an ordering-extension of each and every $D_i \cap R_i$ is sufficient to ensure coherence of D, it is not necessary (Theorem 1).

The second formalization states that D is coherent if and only if there exists no critical loop in D. A critical loop in D is a sequence of at least two ordered pairs (x_μ,y_μ), where μ ranges from 1 to t, such that (i) each (x_μ,y_μ) belongs to some D_{i_μ}, (ii) the second element of each ordered pair is identical to the first element of the succeeding ordered pair, if a succeeding ordered pair exists, (iii) the second element of the last ordered pair is identical to the first element of the first ordered pair and (iv) the set $\{i_1,...,i_t\}$ is not a singleton.[2] We show that, like the first formalization, absence of a critical loop in D is sufficient to ensure coherence of D but is not necessary (Theorem 2).

It is generally believed that these two formalizations are equivalent.[3] We show that this belief is not correct. Indeed the two are logically independent (Theorem 3).

Fortunately, slight modifications in the two formalizations make them equivalent to coherence, and consequently to each other. We show in Theorems 4 and 5 that (i) A rights-assignment is coherent if and only if there exists an ordering-extension of each and every $D_i \cap P(R_i)$, (ii) A rights-assignment D is coherent if and only if there is no modified critical loop in D. A modified critical loop is a sequence of ordered pairs (x_μ, y_μ) which, in addition to satisfying the conditions mentioned in the definition of a critical loop, satisfies the restriction that $x_1,...,x_t$ involved in the loop are all distinct.

We also discuss the important special case of symmetric rights-assignments. For symmetric assignments it is true that D is coherent if and only if there exists an ordering-extension of each and every $D_i \cap R_i$ (Theorem 6).

The analysis of coherence of rights-assignments has some important implications for the way the idea of a liberal individual is formalized. Consider the following definition of a liberal individual.

Let Y be the set of all ordering-extensions of each and every $D_i \cap R_i$. An individual j is defined to be liberal if and only if $R_j^* = R_j \cap R^j$ for some $R^j \in Y$, where R_j^* is the relation that individual j wishes to be counted in social choice.[4]

There are difficulties associated with the above formulation. This formulation does not ensure that R_j^* is a sub-relation of R_j. For instance, if individual j is assigned the ordered pair (x,y) but not (y,x), then if individual j happens to be indifferent between x and y, every ordering-extension belonging to Y would have x preferred to y and consequently we would have x preferred to y in terms of R_j^* as well. Thus, R_j^* may not even faithfully reflect the preferences of individual j over his own protected sphere.[5]

The most important difficulty, however, with this formulation arises because the set of ordering-extensions Y may be empty even though the rights-assignment is coherent, as has been shown in Theorem 1. Consequently, it may be impossible for any one to be liberal in the sense of the above formulation, notwithstanding some individuals' desire to respect other individuals' rights.

Using the result of Theorem 4, we provide a formalization of the idea of a liberal individual which is free from the above difficulties. Our formulation, in addition, has the nice property that, whenever the profile of individual orderings is such that the weak Pareto-criterion and the coherent libertarian condition do not conflict, only ordering-extensions which preserve individuals' unanimous strict preferences are used. Consequently it can be shown that, if there exists at least one liberal individual in the society then, there exists a collective choice rule with unrestricted domain satisfying the coherent libertarian requirement, conditional weak Pareto-criterion and satisfying the property that, whenever the profile of individual orderings is such that the weak Pareto-criterion and the coherent libertarian requirement do not conflict, the chosen elements are Pareto-optimal in the sense of weak Pareto-criterion (Theorem 7).

1. Definitions and Assumptions

We denote the set of social alternatives by X. It is assumed that X is finite and $\# X = m \geq 3$. The finite set of individuals constituting the society is denoted by N and it is assumed that $\# N = n \geq 2$.

Let R be a binary relation on a set X. The asymmetric and symmetric parts of R, to be denoted by P(R) and I(R) respectively, are defined as follows: $\forall x,y \in X : [[(x,y) \in P(R)$ iff $(x,y) \in R$ & $(y,x) \notin R]$ & $[(x,y) \in I(R)$ iff $(x,y) \in R$ & $(y,x) \in R]]$. A binary relation R on a set X is (i) reflexive iff $\forall x \in X : (x,x) \in R$ (ii) connected iff $\forall$ distinct $x,y \in X : [(x,y) \in R \vee (y,x) \in R]$ (iii) transitive iff $\forall x,y,z \in X : [(x,y) \in R$ & $(y,z) \in R \rightarrow (x,z) \in R]$ (iv) an ordering iff it is reflexive, connected and transitive (v) consistent iff $\forall x_1,...,x_t \in X, (t \geq 2) : [(x_1, x_2) \in P(R)$ & $(x_k, x_{k+1}) \in R, k = 2,...,t\text{-}1, \rightarrow (x_t, x_1) \notin R]$ (vi) acyclic iff $\forall x_1,...,x_t \in X, (t \geq 2) : [(x_k, x_{k+1}) \in P(R), k = 1,...,t\text{-}1, \rightarrow (x_1, x_t) \in R]$.

(x,y) $\in$ R will at times be written as xRy. P(R) and I(R) sometimes would be written simply as P and I respectively.

We will assume that each individual i $\in$ N has an ordering R_i over X. We denote the set of all orderings of set X by T. The n-fold Cartesian product of T will be written as T^n. Profiles of individual orderings will be written on the pattern, a = $(R_1^a,...,R_n^a) \in T^n$, or simply as a = $(R_1,...,R_n) \in T^n$.Let R_1 and R_2 be two binary relations on a set X. R_2 is said to be an extension of R_1 (or equivalently R_1 is said to be a sub-relation of R_2) iff (i) $R_1 \subset R_2$ and (ii) $P(R_1) \subset P(R_2)$. If R_2 is an extension of R_1 and R_2 is an ordering, we say that R_2 is an ordering - extension of R_1.

Remark 1 : In a fundamental contribution Suzumura has proved the following theorem :
Theorem (Suzumura) : A binary relation R has an ordering-extension iff R is consistent.[6]

We denote by K the set of all non-empty subsets of X; K = $2^X - \{\emptyset\}$. A choice function C defined over K is a function which, for every S $\in$ K, assigns a unique non-empty subset C(S) of S. A collective choice rule (CCR) f defined over W $\subset T^n$ is a function which, for every profile of individual orderings a = $(R_1^a,...,R_n^a) \in$ W , determines a unique choice function C^a over K ; C^a = f(a).
Unrestricted Domain (U) : A CCR is said to have unrestricted domain iff domain W = T^n.

For every a $\in T^n$ and every S $\in$ K, we define :

$C_p^a(S) = \{x \in S \mid \sim [\exists y \in S : (y,x) \in \bigcap_{i=1}^{n} P(R_i^a)]\}$;

$C_{\bar{p}}^a(S) = \{x \in S \mid \sim [\exists y \in S : (y,x) \in P(\bigcap_{i=1}^{n} R_i^a)]\}$.

Weak Pareto-criterion (P) : A CCR with unrestricted domain satisfies the weak Pareto-criterion iff $\forall$a $\in T^n$ and $\forall$S $\in$ K : $C^a(S) \subset C_p^a(S)$.
Strict Pareto-criterion ($\bar{P}$): A CCR with unrestricted domain satisfies the strict Pareto-criterion iff $\forall$a $\in T^n$ and $\forall$S $\in$ K : $C^a(S) \subset C_{\bar{p}}^a$ (S).

The 2-fold Cartesian product X^2 is the set of all ordered pairs of X. We denote by H the set of all subsets of X^2 ; $H = 2^{X^2}$.

Let $D_i \in H$ ($i \in N$) denote the set of all ordered pairs assigned to individual i, i.e., the protected sphere of individual i. $D = (D_1,...,D_n) \in H^n$ is called the rights-assignment for the society.

For any set X, we define $\Delta_X = \{(x,x) | x \in X\}$.

2. Coherence of Rights-Assignments

Let $D = (D_1,...,D_n)$ be a rights-assignment for the society. For all $a \in T^n$ and for all $S \in K$ we define :
$C^a_D(S) = \{x \in S \mid \sim [\exists y \in S : \exists i \in N : (y,x) \in D_i \cap P(R^a_i)]\}$.

A rights-assignment $D = (D_1,...,D_n)$ is called coherent iff $\forall a \in T^n$ and $\forall S \in K : C^a_D(S) \neq \emptyset$.

In other words, a rights-assignment D is defined to be coherent iff the exercise of rights by itself never results in empty choice set no matter which nonempty subset of alternatives and which profile of individual orderings are considered.

In the social choice literature, the idea of coherence of rights-assignments has been formalized in two different ways. In what follows, we shall show that neither of the two formalizations is equivalent to the coherence of rights- assignments as defined above.
E-coherence: $D = (D_1,...,D_n)$ is defined to be E-coherent iff for every $(R_1,...,R_n) \in$

T^n there exists an ordering-extension R of each and every $D_i \cap R_i$, i = 1,...,n.
L-coherence: Let D = (D_1,...,D_n) be a rights-assignment. A critical loop in D is a sequence of ordered pairs $< (x_\mu, y_\mu) >_{\mu=1}^{t}$ (t $\geq$ 2) such that

(i) $(x_\mu, y_\mu) \in \bigcup_{i=1}^{n} D_i$, $\forall\mu \in \{1,2,...,t\}$

(ii) there exists no $i^* \in$ N such that $(x_\mu, y_\mu) \in D_{i^*}$, $\forall\mu \in \{1,2,...,t\}$

(iii) $x_1 = y_t$ and $x_\mu = y_{\mu-1}$,$\forall\mu \in \{2,...,t\}$.

D is defined to be L-coherent iff there exists no critical loop in D.

Theorem 1 : (a) Coherence of rights-assignment D = (D_1,...,D_n) does not imply E-coherence of D.
(b) E-coherence of D implies coherence of D.
Proof: (a)
Proof consists of an example.
Let X = {x,y,z} , N = {1,2}, D = (D_1,D_2)
$D_1 = X^2 - \{(y,x)\}$
$D_2 = \Delta_X = \{(x,x), (y,y), (z,z)\}$.

Coherence of D is obvious.

Consider the following profile of individual orderings :

1. x I_1 y I_1 z
2. x P_2 y P_2 z

$D_1 \cap R_1 = D_1$
$D_2 \cap R_2 = \Delta_X$

As D_1 is not consistent [we have (x,y) $\in$ P(D_1), (y,z) $\in$ D_1, (z,x) $\in$ D_1], there does not exist any R which is an ordering-extension of each and every $D_i \cap R_i$.

(b)

Suppose D is incoherent. Then, there exists a profile a = (R_1^a,..., R_n^a) $\in$ T^n and an S $\in$ K such that $C_D^a(S)$ is empty. Choose S such that S is the smallest set or one of the smallest sets belonging to K for which $C_D^a(.)$ is empty. Let S = {x_1,...,x_t}, where x_1,...,x_t are all distinct. From the definition of $C_D^a(S)$, it follows that t $\geq$ 2.

As S is one of the smallest sets belonging to K for which $C_D^a(.)$ is empty, it follows that there exists a one-to-one correspondence θ : S $\rightarrow$ S such that :
$\forall x_k \in S : [\exists\, i_k \in N : (\theta(x_k), x_k) \in D_{i_k} \cap P(\,^aR i_k)]$
Define :
$y_1 = x_1$
$y_{k+1} = \theta(y_k)$, k = 1,...,t-1.
We conclude:
(i) $y_k \neq y_{k+1}$, k = 1,...,t-1, and $y_t \neq \theta(y_t)$, as each P (R_i^a) is asymmetric
(ii) y_1,...,y_t are all distinct and $\theta(y_t) = y_1$, as S is one of the smallest sets belonging to K for which $C_D^a(.)$ is empty.
Thus we have,
$\exists j_1,...,j_t \in N$:
$(y_1, y_t) \in Dj_t \cap P(\,^aRj_t)$
$(y_t, y_{t-1}) \in Dj_{t-1} \cap P(\,^aRj_{t-1})$
:
$(y_2, y_1) \in Dj_1 \cap P(\,^aRj_1)$.

Suppose there exists an ordering R which is an extension of each and every $D_i \cap R_i^a$, i = 1,...,n.
For any i $\in$ N and any x,y $\in$ X we have :
$(x,y) \in D_i \cap P(R_i^a) \rightarrow (x,y) \in D_i \cap R_i^a$
$(x,y) \in D_i \cap P(R_i^a) \rightarrow (x,y) \in P(R_i^a)$
$\rightarrow (y,x) \notin R_i^a$
$\rightarrow (y,x) \notin D_i \cap R_i^a$
Therefore,
$(x,y) \in D_i \cap P(R_i^a) \rightarrow (x,y) \in P[D_i \cap R_i^a]$
$\rightarrow (x,y) \in P(R)$

Consequently we must have,
$(y_1, y_t) \in P(R)$, $(y_t, y_{t-1}) \in P(R)$ ... ... $(y_2, y_1) \in P(R)$.
This, however, violates transitivity or asymmetry of P(R). Thus there cannot possibly exist an ordering extension of each and every $D_i \cap R_i^a$, i = 1,...,n.

This establishes the theorem.

Theorem 2: (a) Coherence of rights-assignment $D = (D_1,...,D_n)$ does not imply L-coherence of D.
(b) L-coherence of D implies coherence of D.
Proof: (a)

Proof consists of an example.
Let X = {x,y,z}, N= {1,2}, $D = (D_1, D_2)$
$D_1 = \Delta_X \cup \{(x,y), (y,x)\}$
$D_2 = \Delta_X \cup \{(y,z), (z,y)\}$
Coherence of D is obvious.
Consider the following sequence of ordered pairs $< (x_\mu, y_\mu) >_{\mu=1}^{4}$:
$(x_1, y_1) = (x,y)$
$(x_2, y_2) = (y,z)$
$(x_3, y_3) = (z,y)$
$(x_4, y_4) = (y,x)$
We have,
(i) $(x,y), (y,z), (z,y), (y,x) \in D_1 \cup D_2$
(ii) there exists no $i^* \in N$ such that $(x,y), (y,z), (z,y), (y,x) \in D_{i^*}$
(iii) $x_1 = y_4$, and $x_\mu = y_{\mu-1}$, for all $\mu \in \{2,3,4\}$.
Thus, $(x,y) \in D_1$, $(y,z) \in D_2$, $(z,y) \in D_2$, $(y,x) \in D_1$ is a critical loop in D. This implies that D fails to satisfy L-coherence.

(b)

Suppose D is not coherent. Then there exist $a = (R_1^a,..., R_n^a) \in T^n$ and $S \in K$ such that $C_D^a(S)$ is empty. Choose S such that it is one of the smallest sets belonging to K for which $C_D^a(.)$ is empty. Let $S = \{x_1,...,x_t\}$, where $x_1,...,x_t$ are all distinct. As in the proof of Theorem 1(b), we can conclude that:
$\exists j_1,...,j_t \in N$:

$(y_1, y_t) \in D_{j_t} \cap P({}^aR_{j_t})$
$(y_t, y_{t-1}) \in D_{j_{t-1}} \cap P({}^aR_{j_{t-1}})$
:
$(y_2, y_1) \in D_{j_1} \cap P({}^aR_{j_1})$,
where (i) $y_1,...,y_t$ are all distinct, (ii) $\{y_1,...,y_t\} = \{x_1,...,x_t\}$, (iii) $t \geq 2$.
The above implies :
(i) $(y_1, y_t), (y_t, y_{t-1}),....,(y_2, y_1) \in \bigcup_{i=1}^{n} D_i$
(ii) $\{j_1,...,j_t\}$ is not a singleton, as each R_i^a is an ordering and each $P(R_i^a)$ is asymmetric.

Thus $(y_1, y_t) \in D_{j_t}$, $(y_t, y_{t-1}) \in D_{j_{t-1}}$,....., $(y_2, y_1) \in D_{j_1}$ is a critical loop in D and consequently D is not L-coherent.

Theorems 1 and 2 establish that both formalizations of the idea of coherence are overly strong. While both formalizations rule out incoherent rights-assignments, they rule out some coherent rights-assignments as well. It is generally believed that E-coherence and L- coherence are equivalent. The following theorem shows that this belief is not correct.

Theorem 3: L-coherence and E-coherence are logically independent of each other.
Proof: Proof consists of the following four examples:
(a) Let $X = \{x,y,z\}$, $N = \{1,2\}$, $D = (D_1, D_2)$
$D_1 = \{(x,y), (y,z), (z,y), (x,z), (z,x)\}$
$D_2 = \emptyset$
No critical loop can exist in D, therefore D is L-coherent.
Consider the following profile (R_1, R_2) :
1. xI_1yI_1z
2. xP_2yP_2z
$D_1 \cap R_1 = D_1$
As D_1 is not consistent, there does not exist any R which is an ordering-extension of each and every $D_i \cap R_i$, i = 1,2. Thus D is not E- coherent.
(b) Let $X = \{x,y,z\}$, $N = \{1,2\}$, $D = (D_1, D_2)$
$D_1 = \{(x,y), (y,x)\}$
$D_2 = \{(y,z), (z,y)\}$
$(x,y) \in D_1$, $(y,z) \in D_2$, $(z,y) \in D_2$, $(y,x) \in D_1$ is a critical loop in D. Therefore, D is not L-coherent.

It is clear that for every $(R_1, R_2) \in T^2$, there exists an R which is an ordering-extension of each and every $D_i \cap R_i$, i = 1,2. Thus D is E-coherent.
(c) Let X = {x,y,z}, N = {1,2}, $D = (D_1, D_2)$
$D_1 = \{(x,y)\}$
$D_2 = \{(y,x)\}$.
As $(x,y) \in D_1$, $(y,x) \in D_2$ is a critical loop, it follows that D is not L-coherent. Let (R_1, R_2) be such that xP_1y and yP_2x. For such an (R_1, R_2), no R exists which is an ordering extension of $D_1 \cap R_1$ and $D_2 \cap R_2$. So D is not E-coherent.
(d) Let X = {x,y,z}, N = {1,2}, $D = (D_1,D_2)$
$D_1 = \{(x,y)\}$
$D_2 = \{(y,z)\}$
It is clear that D is both L-coherent and E-coherent.

Now, we shall introduce a modified version of E-coherence and show that the modified version is logically equivalent to coherence.
Modified E-coherence (ME-Coherence): A rights-assignment $D = (D_1,...,D_n)$ is defined to be ME-coherent iff for every $(R_1,...,R_n) \in T^n$ there exists an R such that it is an ordering-extension of each and every $D_i \cap P(R_i)$, i = 1,...,n.

Theorem 4: A rights-assignment $D = (D_1,...,D_n)$ is coherent iff it is ME-coherent.
Proof: First, we establish that ME-coherence implies coherence. Suppose coherence is violated. Then there exist profile $a = (R^a_1,..., R^a_n) \in T^n$ and $S \in K$ such that $C^a_D(S)$ is empty. Choose S such that it is one of the smallest sets belonging to K for which $C^a_D(.)$ is empty. Let $S = \{x_1,...,x_t\}$, where $x_1,...,x_t$ are all distinct. As in the proof of Theorem 1(b), we can conclude that :
$\exists j_1,...,j_t \in N$:
$(y_1, y_t) \in D_{j_t} \cap P(R^a_{j_t})$
$(y_t, y_{t-1}) \in D_{j_{t-1}} \cap P(R^a_{j_{t-1}})$
:
$(y_2, y_1) \in D_{j_1} \cap P(R^a_{j_1})$,
where (i) $y_1,...,y_t$ are all distinct, (ii) $\{y_1,...,y_t\} = \{x_1,...,x_t\}$, (iii) $t \geq 2$.
As for each $i \in N$, $P(R^a_i)$ is asymmetric, it follows that $\forall i \in N$:
$P[D_i \cap P(R^a_i)] = D_i \cap P(R^a_i)$.
Therefore, if there is an R which is an ordering-extension of each and every

$D_i \cap P(R_i^a)$, i $\in$ N, then we must have $(y_1, y_t), (y_t, y_{t-1}), ..., (y_2, y_1) \in P(R)$, which is impossible. Thus D violates ME-coherence.

Next, we show that coherence implies ME-coherence.

For each i $\in$ N, define :

$Q_i^a = D_i \cap P(R_i^a)$ (i)

Let $Q^a = \bigcup_{i=1}^{n} Q_i^a$ (ii)

Take any i $\in$ N and any x,y $\in$ X :

$(x,y) \in Q_i^a \rightarrow (x,y) \in P(R_i^a)$

$\rightarrow (y,x) \notin P(R_i^a)$

$\rightarrow (y,x) \notin Q_i^a$

Therefore,

$P(Q_i^a) = Q_i^a$ (iii)

Consider any x,y $\in$ X :

Suppose $(x,y) \in Q^a$ and $(y,x) \in Q^a$

$(x,y) \in Q^a \rightarrow \exists i \in N : (x,y) \in Q_i^a$

$(y,x) \in Q^a \rightarrow \exists j \in N : (y,x) \in Q_j^a$

As Q_i^a and Q_j^a are asymmetric, it follows that $i \neq j$. Consider $C_D^a(\{x,y\})$. In view of $(x,y) \in Q_i^a$ and $(y,x) \in Q_j^a$, we conclude that it is empty. This, however, contradicts the coherence of D. So we conclude,

$(x,y) \in Q^a \rightarrow (y,x) \notin Q^a$.

This establishes,

$Q^a = P[Q^a]$. (iv)

Next, we show that Q^a is consistent.

Suppose Q^a is not consistent. Then there exist $x_1,...,x_t \in X$, $t \geq 2$, such that :

$(x_1, x_2) \in P(Q^a)$, $(x_k, x_{k+1}) \in Q^a$, k = 2,.....t-1, $(x_t, x_1) \in Q^a$

$\rightarrow \exists i_1,...,i_t \in N : (x_1, x_2) \in Q_{i_1}^a, (x_k, x_{k+1}) \in Q_{i_k}^a$, k = 2,...,t-1, $(x_t, x_1) \in Q_{i_t}^a$

$\rightarrow \exists i_1,...,i_t \in N : (x_1, x_2) \in D_{i_1} \cap P(R_{i_1}^a), (x_k, x_{k+1}) \in D{i_k} \cap P({}^aR{i_k})$, k = 2,...,t-1, $(x_t,x_1) \in D_{i_t} \cap P(R_{i_t}^a)$

$\rightarrow C_D^a(\{x_1,...,x_t\}) = \emptyset$,

which contradicts the coherence of D. This establishes that Q^a is consistent

Therefore, by Suzumura Extension Theorem, there exists an R such that,

$Q^a \subset R$ (v)

and $P(Q^a) \subset P(R)$ (vi)

From (i) - (vi), we conclude that for each i $\in$ N :

$Q_i^a \subset Q^a \subset R$
and $P(Q_i^a) \subset P(Q^a) \subset P(R)$.
Thus, we have shown that there exists an ordering-extension R of each and every $D_i \cap P(R_i^a)$, $i \in N$. This establishes that D is ME-coherent.

Next, we introduce a modified version of L-coherence and show that the modified version is equivalent to coherence.
Modified L-coherence [ML-coherence] : Let D = $(D_1,.....,D_n)$ be a rights-assignment. The sequence $< (x_\mu, y_\mu) \in D_{i_\mu} >_{\mu=1}^{t}$, $i_\mu \in N$, $(t \geq 2)$ is said to be a modified critical loop in D iff (i) $x_\mu = y_{\mu-1}$, $\mu = 2,...,t$ and $x_1 = y_t$ (ii) $\{i_1,...,i_t\}$ contains at least two distinct individuals (iii) $\forall j,k \in \{1,...,t\} : x_j = x_k \rightarrow j = k$. D is defined to be ML-coherent iff there exists no modified critical loop in D.

Theorem 5: A rights assignment D = $(D_1,...,D_n)$ is coherent iff it is ML-coherent.
Proof: Suppose coherence is violated. Then there exist profile a = $(R_1^a,...,R_n^a) \in T^n$ and $S \in K$ such that $C_D^a(S)$ is empty. Choose S such that it is one of the smallest sets belonging to K for which $C_D^a(.)$ is empty. As in the proof of Theorem 1(b), we can conclude that : $\exists j_1,..., j_t \in N$:
$(y_1, y_t) \in D_{j_t} \cap P(R_{j_t}^a)$
$(y_t, y_{t-1}) \in D_{j_{t-1}} \cap P(R_{j_{t-1}}^a)$
:
$(y_2, y_1) \in D_{j_1} \cap P(R_{j_1}^a)$,
where (i) $y_1,...,y_t$ are all distinct, (ii) $\{y_1,...,y_t\}$ = S, (iii) $t \geq 2$. As each $P(R_i^a)$, $i \in N$, is asymmetric and transitive, we conclude that $\{j_1,...,j_t\}$ contains at least two distinct individuals. As $y_1,...,y_t$ are all distinct, it follows that $(y_1, y_t) \in D_{j_t}$, $(y_t, y_{t-1}) \in D_{j_{t-1}}$,..., $(y_2, y_1) \in D_{j_1}$ $(t \geq 2)$ is a modified critical loop in D. Therefore D is not ML-coherent. This establishes that ML-coherence implies coherence.

Next we show that coherence implies ML-coherence.
Suppose D does not satisfy ML-coherence. Then, there exists a modified critical loop in D, i.e., there exist $x_1,...,x_t \in X$, $i_1,...,i_t \in N$, $t \geq 2$, such that (i) $(x_k,x_{k+1}) \in D_{i_k}$, k = 1,...,t-1 and $(x_t, x_1) \in D_{i_t}$, (ii) $\{i_1,...,i_t\}$ contains at least two distinct individuals and (iii) $x_1,...,x_t$ are all distinct.

Let a $\in$ T^n be such that $(x_k, x_{k+1}) \in P({}^aR_{i_k})$, k = 1,...,t-1, $(x_t, x_1) \in P(R^a_{i_t})$. Such an a $\in$ T^n can always be found because (i) $\{i_1,...,i_t\}$ contains at least two distinct individuals and (ii) $x_1,...,x_t$ are all distinct. As $(x_k, x_{k+1}) \in D_{i_k} \cap P(R^a_{i_k})$, k = 1,...,t-1, and $(x_t, x_1) \in D_{i_t} \cap P(R^a_{i_t})$, it follows that $C^a_D(\{x_1,...,x_t\})$ is empty, which violates coherence of D. This establishes the theorem.

3. Symmetric Rights-assignments

Now, we consider the important special case of symmetric rights-assignments. A rights-assignment D = $(D_1,...,D_n)$ is defined to be symmetric iff $\forall i \in N : [\forall x,y \in X : [(x,y) \in D_i$ iff $(y,x) \in D_i]]$. It turns out that a necessary and sufficient condition for a symmetric rights-assignment to be coherent is that it satisfies E-coherence.

Theorem 6 : A symmetric rights-assignment D = $(D_1,...,D_n)$ is coherent iff it is E-coherent.
Proof: E-coherence implies coherence, irrespective of whether D is symmetric or not, has already been proved in Theorem 1. Suppose D is coherent. Consider any a = $(R^a_1,...,R^a_n) \in T^n$. Define,

$G^a_i = D_i \cap R^a_i, \quad i \in N$ (i)

$G^a = \bigcup_{i=1}^{n} G^a_i$ (ii)

From (i) and (ii), we obtain,

$G^a_i \subset G^a, \ i \in N$ (iii)

Take any i $\in$ N and any x,y $\in$ X :

$(x,y) \in P(G^a_i) \rightarrow (x,y) \in P(R^a_i)$, as D is symmetric (iv)

$\rightarrow x \neq y$ (v)

$(x,y) \in P(G^a_i)$ & $(y,x) \in G^a \rightarrow (x,y) \in D_i$ & $(y,x) \in D_j$ & $i \neq j$ (vi)

(v) and (vi) imply that $(x,y) \in D_i$, $(y,x) \in D_j$ is a modified critical loop, which contradicts the coherence of D in view of Theorem 5. Therefore, we conclude

$(x,y) \in P(G^a_i) \rightarrow (x,y) \in P(G^a)$

Thus for every i $\in$ N :

$P(G^a_i) \subset P(G^a)$. (vii)

Next, we show that G^a is consistent. Suppose G^a is not consistent. Then there exist $y_1,...,y_v \in X$, $v \geq 2$, such that $(y_1, y_2) \in P(G^a)$, $(y_k, y_{k+1}) \in G^a$, k = 2,...,v-1, $(y_v, y_1) \in G^a$. As $P(G^a)$ is asymmetric, it follows that there exist $x_1,...,x_t \in X$, $t \geq 2$, with $x_1,...,x_t$ all distinct, such that $(x_1, x_2) \in P(G^a)$, $(x_k, x_{k+1}) \in G^a$, k = 2,...,t-1, $(x_t, x_1) \in G^a$. Therefore, $\exists i_1,...,i_t \in N : (x_1, x_2) \in D_{i_1} \cap P(R^a_{i_1})$, $(x_k, x_{k+1}) \in D_{i_k} \cap R^a_{i_k}$, k = 2,...,t-1, $(x_t, x_1) \in D_{i_t} \cap R^a_{i_t}$. As for each $i \in N$, R^a_i is an ordering and $P(R^a_i)$ is asymmetric, it follows that $\{i_1,...,i_t\}$ contains at least two distinct individuals. This establishes that $(x_1, x_2) \in D_{i_1}$, $(x_k, x_{k+1}) \in D_{i_k}$, k = 2,...,t-1, $(x_t, x_1) \in D_{i_t}$ is a modified critical loop in D, contradicting coherence of D. This establishes that G^a is consistent. Therefore, by Suzumura Extension Theorem, it follows that there exists an ordering-extension R of G^a, i.e.,

$G^a \subset R$ (viii)

and $P(G^a) \subset P(R)$ (ix)

(iii), (vii), (viii) and (ix) establish that for each $i \in N$:

$G^a_i \subset G^a \subset R$

and $P(G^a_i) \subset P(G^a) \subset P(R)$,

which proves the E-coherence of D.

4. Liberal Individuals and Resolution of Sen Paradox

An individual who respects the rights of other individuals will be called a liberal. In what follows we provide a formalization of the idea of an individual being liberal. Let $D = (D_1,...,D_n)$ be a coherent rights-assignment and let $a = (R^a_1,..., R^a_n) \in T^n$ be a profile of individual orderings. By Theorem 4, there exists an R which is an ordering-extension of each and every $D_i \cap P(R^a_i)$, $i \in N$. Let Z denote the set of all ordering-extensions of each and every $D_i \cap P(R^a_i)$, $i \in N$. Let

$\Pi = \{(x,y) \in X^2 | \forall i \in N : xP(R^a_i)y\}$.

Define for each $R \in Z$:

$p(R) = \# [P(R) \cap \Pi]$.

We define $Z_p \subset Z$ by

$Z_p = \{R \in Z | \forall R' \in Z : p(R) \geq p(R')\}$.

Let R^*_j, $j \in N$, denote the sub-relation of R^a_j that the individual j wants to be counted in social choice.

We define individual j to be liberal iff for some $R^j \in Z_p$, $P(R^*_j) = P(R^a_j) \cap P(R^j)$ and $I(R^*_j) = I(R^a_j) \cap I(R^j)$

The set of liberal individuals will be denoted by N_L.
It will be assumed that $\forall i \in N - N_L : R_i^* = R_i^a$.
For every $(R_1^*,...,R_n^*)$, corresponding to $(R_1^a,...,R_n^a) = a \in T^n$, and every $S \in K$ we define,

$$C_p^a*(S) = \{x \in S \mid \sim [\exists y \in S : (y,x) \in \bigcap_{i \in N} P(R_i^*)]\}$$

$$C_{\bar{p}}^a*(S) = \{x \in S \mid \sim [\exists y \in S : (y,x) \in P(\bigcap_{i \in N} R_i^*)]\}$$

Conditional weak Pareto-criterion (CWP) : A CCR with unrestricted domain satisfies CWP iff $\forall a \in T^n$ and $\forall S \in K : C^a(S) \subset C_p^a*(S)$.
Conditional Pareto-criterion (CP) : A CCR with unrestricted domain satisfies CP iff
$\forall a \in T^n$ and $\forall S \in K : C^a(S) \subset C_{\bar{p}}^a*(S)$.

Lemma 1: Let $D = (D_1,...,D_n)$ be a coherent rights-assignment and $a = (R_1^a,...,R_n^a) \in T^n$ a profile of individual orderings. If a is such that for each $S \in K$, $C_D^a(S) \cap C_p^a(S) \neq \emptyset$, then there exists an R which is an ordering-extension of each and every $D_i \cap P(R_i^a)$, $i \in N$ as well as of Π.
Proof: Let $a = (R_1^a,...,R_n^a)$ be a profile of individual orderings such that $\forall S \in K : C_D^a(S) \cap C_p^a(S) \neq \emptyset$. Define,

$$Q_i^a = D_i \cap P(R_i^a), \ i \in N \qquad \text{(i)}$$

$$Q^a = \bigcup_{i=1}^{n} Q_i^a \qquad \text{(ii)}$$

$$J = Q^a \cup \Pi \qquad \text{(iii)}$$

By definition Π as well as each Q_i^a, $i \in N$, is asymmetric. Q^a is also asymmetric as has been shown in Theorem 4. We now show that J is asymmetric. Suppose $(x,y) \in J$ and $(y,x) \in J$, $x,y \in X$. In view of asymmetry of Π and Q^a there are only two possibilities to be considered : (i) $(x,y) \in Q^a$ & $(y,x) \in \Pi$, (ii) $(x,y) \in \Pi$ & $(y,x) \in Q^a$. $(x,y) \in Q^a$ implies that $\exists j : xP(R_j^a)y$ and $(y,x) \in \Pi$ implies that $\forall i \in N : yP(R_i^a)x$. Therefore (i) is not possible. By a similar argument (ii) is not possible.

Next, we show that J is consistent. Suppose not. Then there exist $x_1,...,x_t$, $t \geq 2$, such that $(x_1, x_2) \in P(J)$, $(x_k, x_{k+1}) \in J$, $k = 2,...,t-1$, $(x_t, x_1) \in J$. As Q^a is consistent by Theorem 4 and Π is transitive, it follows that J being nonconsistent implies that $C^a_D(S) \cap C^a_p(S) = \emptyset$, where $S = \{x_1,...,x_t\}$. This, however, contradicts the choice of $a = (R^a_1,..., R^a_n)$. So J must be consistent. Therefore there exists an ordering-extension R of J.

From definitions (i) - (iii), asymmetry of Q^a_i, $i \in N$, Q^a, Π and J, and the fact that R is an ordering-extension of J, we conclude :

$Q^a_i \subset Q^a \subset J \subset R$, $i \in N$
$P(Q^a_i) \subset P(Q^a) \subset P(J) \subset P(R)$, $i \in N$
$\Pi \subset J \subset R$
$P(\Pi) \subset P(J) \subset P(R)$, which establishes the lemma.
D-Libertarianism (DL) : Let $D = (D_1,...,D_n)$ be any given coherent rights-assignment. A CCR satisfies DL iff $\forall a \in T^n$ and $\forall S \in K : C^a(S) \subset C^a_D(S)$.

We say that for $a = (R^a_1,...,R^a_n) \in T^n$, weak Pareto- criterion and D-Libertarianism do not conflict iff $\forall S \in K : C^a_D(S) \cap C^a_p(S) \neq \emptyset$.

Theorem 7: Let $D = (D_1,...,D_n)$ be any given coherent rights-assignment. If there is at least one liberal individual in the society then there exists a CCR f satisfying (i) U, (ii) DL, (iii) CWP and (iv) the property that for every $a \in T^n$ for which WP and DL do not conflict, $\forall S \in K : C^a(S) \subset C^a_D(S) \cap C^a_p(S)$, where $C^a = f(a)$.
Proof: Consider any profile $a = (R^a_1,...,R^a_n) \in T^n$ and let $(R^*_1,...,R^*_n)$ correspond to a in the manner defined above.
Define :
$R_0 = \{(x,y) \in X^2 \mid (y,x) \notin P^* \cup P\}$,
where $P^* = \bigcap_{i \in N} P(R^*_i)$ and $P = \bigcap_{j \in N_L} P(R^j)$, $R^j \in Z_p$.

Take any $j \in N_L$. Then $P(R^*_j) \subset P(R^j)$ (A)
Therefore, $P^* \cup P \subset P(R^j)$ (B)
From (B) we conclude that $P^* \cup P$ is asymmetric and consequently R_0 is reflexive and connected. Take any $x,y \in X$. $(x,y) \in P(R_0) \rightarrow (y,x) \notin R_0$, which in turn implies $(x,y) \in P^* \cup P$. On the other hand $(x,y) \in P^* \cup P$ implies $(y,x) \notin R_0$ which in turn implies $(x,y) \in P(R_0)$, as R_0 has already been shown to be reflexive and connected. This establishes that $P^* \cup P = P(R_0)$. As a $P^* \cup P$ cycle would imply, in view of (B), a $P(R^j)$ cycle negating transitivity of R^j or asymmetry of $P(R^j)$, we conclude that no $P^* \cup P$ cycle is possible. As $P(R_0) = P^* \cup P$, it follows that no $P(R_0)$ cycle is possible which in view of reflexivity and connectedness of R_0 establishes that R_0 is acyclic.

Now define for every $S \in K : C^a(S) = \{x \in S \mid \forall y \in S : xR_0y\}$. In view of the finiteness of X and the fact that R_0 is reflexive, connected and acyclic, it follows from Sen's theorem[7] that C^a is a well defined choice function. Now define a CCR f by : $\forall a \in T^n : f(a) = C^a$.

(i) By construction f satisfies U.

(ii)Take any $a = (R_1^a,...,R_n^a) \in T^n$ and any $S \in K$. Suppose $y \notin C_D^a(S)$. Then $y \notin S$; or there exist $x \in S$, $i \in N : (x,y) \in D_i \cap P(R_i^a)$. As $D_i \cap P(R_i^a) = P[D_i \cap P(R_i^a)]$,we conclude that $y \notin S$ or $(x,y) \in P \subset P^* \cup P = P(R_0)$. Consequently $y \notin C^a(S)$, i.e., $C^a(S) \subset C_D^a(S)$. Thus f satisfies DL.

(iii) Take any $a = (R_1^a,...,R_n^a) \in T^n$ and any $S \in K$. Suppose $y \notin C_{p^*}^a(S)$. Then $y \notin S$ or $\exists x \in S : \forall i \in N : (x,y) \in P(R_i^*)$. That is to say, $y \notin S$ or $(x,y) \in P^* \subset P^* \cup P = P(R_0)$. Consequently $y \notin C^a(S)$, i.e., $C^a(S) \subset C_{p^*}^a(S)$. Thus f satisfies CWP.

(iv) Consider any $a = (R_1^a,...,R_n^a) \in T^n$ such that $\forall S \in K : C_D^a(S) \cap C_p^a(S) \neq \emptyset$, and any $S \in K$. By Lemma 1, there exists an R which is an ordering-extension of each and every $D_i \cap P(R_i^a)$, $i \in N$, and of Π. Therefore, we conclude that every $R \in Z_p$ is an ordering-extension of each and every $D_i \cap P(R_i^a)$, $i \in N$ and of Π. Suppose $y \notin C_p^a(S)$. Then $y \notin S$ or $\exists x \in S : \forall i \in N : (x,y) \in P(R_i^a)$. This implies that $y \notin S$, or $(x,y) \in \Pi$ and consequently $(x,y) \in P(R^j)$, $\forall j \in N_L$. Therefore $y \notin S$ or $(x,y) \in P \subset P^* \cup P = P(R_0)$. So $y \notin C^a(S)$. In view of (ii), we conclude that $C^a(S) \subset C_D^a(S) \cap C_p^a(S)$.

This establishes the theorem.[8]

References

1. Arrow, K.J. (1951): Social Choice and Individual Values, Second edition 1963, Wiley, New York.
2. Farrell, M.J. (1976): Liberalism in the theory of social choice, Review of Economic Studies, 43, 3-10.
3. Gibbard, A. (1974): A Pareto-consistent libertarian claim, Journal of Economic Theory, 7, 388-410.
4. Sen, A.K. (1970): Collective Choice and Social Welfare, Holden-Day, San Francisco, and Oliver and Boyd, Edinburgh, Reprinted North-Holland, Amsterdam.
5. Sen, A.K. (1970): The impossibility of a Paretian liberal, Journal of Political Economy, 78, 152-7.
6. Sen, A. K. (1976) : Liberty, unanimity and rights, Economica, 43, 217-45.
7. Suzumura, K. (1978): On the consistency of libertarian claims, Review of Economic Studies, 45, 329-42.
8. Suzumura, K. (1983): Rational Choice, Collective Decisions and Social Welfare, Cambridge University Press, Cambridge.
9. Wriglesworth, J.L. (1985): Libertarian Conflicts in Social Choice, Cambridge University Press, Cambridge.

Notes

1. See Suzumura [7].
2.The idea of a critical loop was introduced by Farrell [2]. The definition of critical loop given here is that of Suzumura [7]. Critical loop in the sense of Suzumura, however, is not equivalent to critical loop in the sense of Farrell. Both Farrell and Suzumura definitions differ from the definition of modified critical loop introduced in this paper.
3. See Lemma 1 in Suzumura [7].
4. See [7] and [8].
5. The important distinction between a person's preferences R_j and the preferences that he would like to be counted in social choice R_j^* was introduced by Sen [6]. Sen requires R_j^* to be a sub-relation of R_j.
6. See Suzumura [8].
7. See Sen [4].
8. In [6] Sen has shown that if there exists at least one liberal individual in the society then there exists a CCR satisfying unrestricted domain, coherent libertarianism (CL) and the conditional strict Pareto-criterion (CP). Under the assumption that there exists a liberal individual in the society, Suzumura [7] has proved the existence of a CCR satisfying conditions U, CL and CP. Suzumura uses the formalization of the idea of a liberal individual discussed in the introduction part of this paper. By a slight modification in the proof of Theorem 7, one can show the existence of a CCR satisfying CP in addition to the conditions mentioned in the statement of the Theorem, provided there is at least one individual in the society who is liberal in the sense of this paper.