CONFORMISM, NONCONFORMISM AND VOTING EQUILIBRIA

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ABSTRACT

This paper is an attempt to investigate the consequences for voting equilibria when some individuals do not possess intrinsic preferences and instead endeavour to relate their preferences to social preferences by conforming or nonconforming. Nonconformist behaviour (desire to have preferences which are opposite or converse of social preferences) on the part of some individuals can lead to nonexistence of equilibrium. Conformist behaviour (desire to have the same preferences as social preferences) can lead to multiple equilibria; in particular, it can lead to two mutually converse linear orderings of social alternatives emerging as equilibria.

For the class of neutral and monotonic binary social decision rules we show that : (i) the possibility of nonconformist behaviour leading to nonexistence of equilibrium exists if and only if the social decision rule is not a simple game, and (ii) the possibility of conformist behaviour leading to two mutually converse linear orderings of social alternatives emerging as equilibria exists if and only if the social decision rule is non-null.

For the method of majority decision we obtain a complete characterization of situations (i) corresponding to which there is no equilibrium, and (ii) corresponding to which two mutually converse linear orderings of social alternatives are equilibria.

Key Words : Conformism, Nonconformism, Voting Equilibria, Simple Games, The Method of Majority Decision.
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In the standard social choice-theoretic framework, it is assumed that individuals are autonomous and have intrinsic preferences over social alternatives. Individual preferences determine social preferences in accordance with the social decision rule adopted by the society. Unless one considers society as merely a collection of individuals, the very idea of society implies that social preferences would play a significant role in the formation of individual preferences. In diverse collective contexts, behavioural patterns exist which are indicative of at least a section of individuals not possessing intrinsic preferences. For instance, in the context of voting behaviour in democracies, existence of attitudes such as `being anti-establishment', `not wanting to waste one vote', `supporting the underdog' etc., implies that not all individuals can be considered as autonomous. Unless one hypothesizes frequent changes in the intrinsic preferences of autonomous individuals, it would be difficult to explain the extraordinary rapidity with which collective preferences change in some social contexts, e.g., changes in fashions, changes in what are considered to be `in' things etc., without assuming that at least some individuals do not have intrinsic preferences and simply react to what they perceive as social preferences.

This paper is a preliminary attempt to look at some logical implications of the assumption, that not all individuals have intrinsic preferences over social alternatives, i.e., that not all individuals are autonomous. Non-autonomous individuals do not have intrinsic preferences, instead they relate their preferences to social preferences. In this paper we focus on the two polar cases where relating to social preferences either means desiring to have the same preferences as what are perceived to be social preferences (conformist behaviour) or desiring to have preferences which are converse or opposite of what are perceived to be social preferences (nonconformist behaviour).

The social binary weak preference relation (BWPR) is determined by individual BWPRs of all individuals, autonomous and non-autonomous, in accordance with the social decision rule chosen by the society. On the other hand, the BWPRs of non-autonomous individuals are determined by the social BWPR. Consequently, for a given profile of autonomous individuals' BWPRs, there will be equilibrium iff the social decision rule generates a social BWPR R corresponding to the profile of individual BWPRs where every conformist's BWPR is R, every nonconformist's BWPR is $\overline{R}$, $\overline{R}$ being the converse relation of R, and autonomous individuals have BWPRs as in the given profile.

If it is assumed that all individuals, autonomous and non-autonomous, have orderings over the set of social alternatives, then the notion of equilibrium needs to be reformulated as social BWPR can be non-transitive. The most appropriate reformulation seems to be to define a social BWPR R to be equilibrium, corresponding to a given profile of autonomous individuals' orderings, iff R is yielded by the social decision rule corresponding to the given profile of autonomous individuals' orderings, all conformists' orderings being transitive closure of R, and all nonconformists' orderings being transitive closure of $\overline{R}$.

If one ignores the strategic aspects, i.e., assumes that every one votes sincerely then, if all individuals are autonomous then there is always a unique equilibrium. The social BWPR determined by the social decision rule corresponding to the given profile of individual BWPRs constitutes the unique equilibrium for the given profile of individual BWPRs. However, nonconformist behaviour can lead to nonexistence of any equilibrium; and conformist behaviour can lead to multiple equilibria. In particular, conformist behaviour can result in both L and $\overline{L}$ emerging as equilibria, where L is some linear ordering of the set of social alternatives and $\overline{L}$ is the converse relation of L.

Consider majority voting when the set of social alternatives consists of x and y, and the society comprises of n = 2k, k $\geq$ 2, individuals, n-1 autonomous individuals and 1 nonconformist. Suppose k of the autonomous individuals prefer x to y and the remaining k-1 autonomous individuals prefer y to x. Then, corresponding to this profile of preferences of autonomous individuals there is no equilibrium. If we consider majority voting, when the set of social alternatives is {x,y}, and the society consists of 2k, k $\geq$ 1, autonomous individuals and 1 conformist, then if k autonomous individuals prefer x to y and the remaining k autonomous individuals prefer y to x then both xPy and yPx emerge as equilibria. In the context of voting in democracies, the absence of equilibrium can manifest itself in `political cycles' where a small number of political parties (usually two) take turns in winning elections. On the other hand, if in some situation both L and $\overline{L}$ are equilibria, where L is some linear ordering of social alternatives, then relating social decision to `social verdict' or `social will' becomes somewhat problematic.

In this paper we investigate the implications of non-autonomous behaviour in the context of neutral and monotonic binary social decision rules. We show that if f is a neutral and monotonic binary social decision rule then there do not exist a decomposition of the set of individuals among autonomous individuals, conformists and nonconformists, and a profile of autonomous individuals' orderings such that there is no equilibrium iff f is a simple game. The proposition is valid with either of the two definitions of equilibrium (theorems 1 and 2). The proposition holds even if it is assumed that there is only one non-autonomous individual in the society who is a nonconformist (theorems 4 and 5).

If f is a neutral and monotonic binary social decision rule then there do not exist a decomposition of the set of individuals among autonomous individuals, conformists and nonconformists, a profile of autonomous individuals' orderings and a linear ordering L of the set of social alternatives such that both L and $\overline{L}$ are equilibria iff f is null (theorem 3). This proposition, however, does not hold if one assumes that there is a single non-autonomous individual in the society who is a conformist. Theorem 6 provides the characterization with this restriction on the class of decompositions.

In theorems 7 and 8 we obtain complete characterization of the profiles of autonomous individuals' orderings corresponding to which there is no equilibrium under the method of majority decision. For the method of majority decision, theorem 9 provides a complete characterization of profiles of autonomous individuals' orderings corresponding to which, for some linear ordering L of social alternatives, both L and $\overline{L}$ are equilibria. In theorem 10 it is shown that, for the subclass of binary social decision rules satisfying the conditions of neutrality, monotonicity and anonymity, a sufficient condition for nonexistence of a profile of autonomous individuals' orderings and a linear ordering L of social alternatives such that both L and $\overline{L}$ are equilibria is that the number of nonconformists be at least as large as the number of conformists. Combining theorems 1 and 10 we obtain the important proposition that if the number of nonconformists is at least as large as the number of conformists then neither the problem associated with nonconformism (nonexistence of equilibrium) nor the problem associated with conformism (both L and $\overline{L}$ being equilibria for some linear ordering L of social alternatives) can arise if the social decision rule is an anonymous simple game.

1. Notation and Definitions

The set of social alternatives and the set of individuals constituting the society are denoted by S and N respectively. We assume S and N to be finite. We denote #S and #N by s and n respectively; and assume $s \geq 2$, $n \geq 2$. Each individual $i \in N$ is assumed to have a binary weak preference relation R_i on S. We denote asymmetric parts of binary relations R_i, R_i^o, R_i^*, R, R^o, R^* etc., by P_i, P_i^o, P_i^*, P, P^o, P^* etc., respectively; and symmetric parts by I_i, I_i^o, I_i^*, I, I^o, I^* etc., respectively.

We define a binary relation R on a set S to be (i) reflexive iff $(\forall x \in S)$ (xRx), (ii) connected iff $(\forall x,y \in S)$ $(x \neq y \rightarrow xRy \vee yRx)$, (iii) antisymmetric iff $(\forall x,y \in S)$ $(xRy \wedge yRx \rightarrow x = y)$, (iv) transitive iff $(\forall x,y,z \in S)$ $(xRy \wedge yRz \rightarrow xRz)$, (v) an ordering iff it is reflexive, connected and transitive, and (vi) a linear ordering iff it is reflexive, connected, anti-symmetric and transitive.

Corresponding to a binary relation R on a set S, we define the opposite or converse relation, to be denoted by $\overline{R}$, by : $(\forall x,y \in S)$ $(x\overline{R}y \leftrightarrow yRx)$.

We denote by C the set of all reflexive and connected binary relations on S and by T the set of all reflexive, connected and transitive binary relations (orderings) on S. A social decision rule (SDR) f is a function from $Z \subseteq C^n$ to C; $f : Z \mapsto C$. Throughout this paper, we assume $T^n \subseteq Z \subseteq C^n$. The social binary weak preference relations corresponding to $(R_1,...,R_n)$, $(R'_1,...,R'_n)$ etc., will be denoted by R, R′ etc., respectively.

An SDR satisfies binariness or independence of irrelevant alternatives (I) iff $(\forall (R_1,...,R_n), (R'_1,...,R'_n) \in Z)$ $(\forall x,y \in S)$ $[(\forall i \in N)\, [(xR_iy \leftrightarrow xR'_iy) \wedge (yR_ix \leftrightarrow yR'_ix)] \rightarrow [(xRy \leftrightarrow xR'y) \wedge (yRx \leftrightarrow yR'x)]]$, and (ii) monotonicity (M) iff $(\forall (R_1,...,R_n), (R'_1,...,R'_n) \in Z)$ $(\forall x \in S)$ $[(\forall i \in N)\, [(\forall a,b \in S - \{x\})\, (aR_ib \leftrightarrow aR'_ib) \wedge (\forall y \in S - \{x\})\, [(xP_iy \rightarrow xP'_iy) \wedge (xI_iy \rightarrow xR'_iy)]] \rightarrow (\forall y \in S - \{x\})\, [(xPy \rightarrow xP'y) \wedge (xIy \rightarrow xR'y)]]$.

Let Φ be the set of all permutations of the set of alternatives S. Let $\phi \in \Phi$. Corresponding to a binary relation R on a set S, we define the binary relation $\phi(R)$ on S by; $(\forall x,y \in S)$ $[\phi(x)\, \phi(R)\, \phi(y) \leftrightarrow xRy]$. An SDR satisfies neutrality (N) iff $(\forall (R_1,...,R_n), (R'_1,...,R'_n) \in Z)$ $[(\exists \phi \in \Phi)\, (\forall i \in N)\, [R'_i = \phi(R_i)] \rightarrow R' = \phi(R)]$.

It is clear from the definitions of conditions I, M and N that an SDR f : Z $\mapsto$ C, $T^n \subseteq Z \subseteq C^n$, satisfying condition I satisfies (i) neutrality iff $(\forall(R_1,...,R_n), (R'_1,...,R'_n) \in Z)$ $(\forall x,y,z,w \in S)$ $[(\forall i \in N)$ $[(xR_iy \leftrightarrow zR'_iw) \wedge (yR_ix \leftrightarrow wR'_iz)] \rightarrow [(xRy \leftrightarrow zR'w) \wedge (yRx \leftrightarrow wR'z)]]$, and (ii) monotonicity (M) iff $(\forall(R_1,...,R_n), (R'_1,...,R'_n) \in Z)$ $(\forall x,y \in S)$ $[(\forall i \in N)$ $[(xP_iy \rightarrow xP'_iy) \wedge (xI_iy \rightarrow xR'_iy)] \rightarrow [(xPy \rightarrow xP'y) \wedge (xIy \rightarrow xR'y)]]$.

Let Π be the set of all permutations of the set of positive integers {1,2,...,n}. An SDR satisfies the condition of anonymity (A) iff $(\forall(R_1,...,R_n), (R'_1,...,R'_n) \in Z)$ $[(\exists p \in \Pi)$ $(\forall i \in N)$ $(R'_i = R_{p(i)}) \rightarrow R = R']$.

$V \subseteq N$ is defined to be winning iff $(\forall(R_1,...,R_n) \in Z)$ $(\forall x,y \in S)$ $[(\forall i \in V)$ $(xP_iy) \rightarrow xPy]$. We denote by W the set of all winning subsets of N. $V \subseteq N$ is minimally winning iff it is winning and no proper subset of it is winning. The set of all minimally winning subsets of N will be denoted by W_m. Let $M \subset N$. We define $V \subseteq N - M$ to be (N − M)-winning iff $(\forall(R_1,...,R_n) \in Z)$ $(\forall x,y \in S)$ $[(\forall i \in M)$ $(xI_iy) \wedge (\forall i \in V)$ $(xP_iy) \rightarrow xPy]$. $V \subseteq N - M$ is minimally (N − M)-winning iff it is (N − M)-winning and no proper subset of it is (N − M)-winning.

Remark 1 : Let f be an SDR and $M \subset N$. If $V_1 \subseteq N - M$ and $V_2 \subseteq N - M$ are (N − M)-winning then $V_1 \cap V_2$ must be nonempty, as $V_1 \cap V_2 = \emptyset$ would lead to a contradiction if we have for $x,y \in S$, $[(\forall i \in M)$ $(xI_iy) \wedge (\forall i \in V_1)$ $(xP_iy) \wedge (\forall i \in V_2)$ $(yP_ix)]$ entailing $(xPy \wedge yPx)$.

Remark 2 : Let $V \subseteq N - M$ be (N − M)-winning, $M \subset N$. Then by the finiteness of V and the fact that the empty set can never be (N − M)-winning, it follows that there exists a nonempty $V' \subseteq V$ such that V' is minimally (N − M)-winning.

A social decision rule is called (i) null iff $(\forall(R_1,...,R_n) \in Z)$ $(\forall x,y \in S)$ (xIy), and (ii) a simple game iff $(\forall(R_1,...,R_n) \in Z)$ $(\forall x,y \in S)$ $[xPy \leftrightarrow (\exists V \in W)$ $(\forall i \in V)$ $(xP_iy)]$. The method of majority decision (MMD) is defined by : $(\forall(R_1,...,R_n) \in Z)$ $(\forall x,y \in S)$ $[xRy \leftrightarrow \#\{i \in N \mid xP_iy\} \geq \#\{i \in N \mid yP_ix\}]$.

Let $S_1 \subseteq S$ and let R be a binary relation on S. We define restriction of R to S_1, denoted by $R|S_1$, by $R|S_1 = R \cap (S_1 \times S_1)$. Let $C_1 \subseteq C$. We define restriction of C_1 to S_1, denoted by $C_1|S_1$, by $C_1|S_1 = \{R|S_1 \mid R \in C_1\}$.

Let R_1 and R_2 be binary relations on a set S. We define composition of R_1 and R_2, denoted by R_1R_2, by $(\forall x,y \in S)$ $[xR_1R_2y \leftrightarrow (\exists z \in S)$ $(xR_1z \wedge zR_2y)]$.

Let R be a binary relation on a set S. We define : $[R^1 = R;\ R^k = RR^{k-1},\ k \geq 2]$. Transitive closure of binary relation R, denoted by t(R), is defined by : $t(R) = \bigcup_{k=1}^{\infty} R^k$. For any binary relation R we denote asymmetric and symmetric parts of its transitive closure by P(t(R)) and I(t(R)) respectively.

Let N be a set. $(N_1,...,N_m)$ is called a decomposition of N iff (i) $\bigcup_{k=1}^{m} N_k = N$ and (ii) $N_j \cap N_k = \emptyset$, $j \neq k$; j,k = 1,...,m.

We denote by A, H and D the set of autonomous individuals, the set of conformists and the set of nonconformists respectively. #A, #H and #D are denoted by a, h and d respectively.

Let $f : C^n \mapsto C$ be a social decision rule. Let (A,H,D) be a decomposition of the set of individuals N, $A = \{j_1,...,j_a\}$. Let $(R_{j_1},...,R_{j_a})$ be a profile of autonomous individuals' orderings. Then, $R \in C$ is an equilibrium iff $R = f(R_{j_1},...,R_{j_a}, (\forall i \in H)\ (R_i = R), (\forall i \in D)\ (R_i = \overline{R}))$.

The above definition of equilibrium is inapplicable if the domain of the SDR is taken to be T^n. For social decision rules with domain T^n, the most appropriate way to define equilibrium seems to be as given below.

Let $f : T^n \mapsto C$ be an SDR. Let (A,H,D) be a decomposition of the set of individuals N, $A = \{j_1,...,j_a\}$. Let $(R_{j_1},...,R_{j_a})$ be a profile of autonomous individuals' orderings. Then, $R \in C$ is an equilibrium iff $R = f(R_{j_1},...,R_{j_a}, (\forall i \in H)\ (R_i = t(R)), (\forall i \in D)\ (R_i = t(\overline{R})))$.

Remark 3 : For any binary relation R on a set S, we have : $\overline{t(R)} = t(\overline{R})$.
Proof: Let $x,y \in S$.
$x\overline{t(R)}y \leftrightarrow yt(R)x$
$\leftrightarrow$ ($\exists$ a positive integer k) (yR^kx)
$\leftrightarrow$ ($\exists$ a positive integer k) $(\exists z_1,...,z_{k-1} \in S)$ $(yRz_1 \wedge z_1Rz_2 \wedge ... \wedge z_{k-1}Rx)$

$\leftrightarrow$ ($\exists$ a positive integer k) ($\exists z_1,...,z_{k-1} \in S$) ($x\overline{R}\, z_{k-1} \wedge ... \wedge z_2\overline{R}\, z_1 \wedge z_1\overline{R}\, y$)
$\leftrightarrow$ ($\exists$ a positive integer k) ($x(\overline{R}\,)^k y$)
$\leftrightarrow$ $xt(\overline{R}\,)y$.

Therefore, it is immaterial whether one takes $(\forall i \in D)$ $(R_i = t(\overline{R}\,))$ or $(\forall i \in D)$ $(R_i = \overline{t(R)})$ in the definition of equilibrium.

2. Characterization Theorems

Theorem 1 : Let $f : C^n \mapsto C$ be a neutral and monotonic binary social decision rule. Then, for every decomposition (A,H,D) of the set of individuals N, $A = \{j_1,j_2,...,j_a\}$, and for every profile of orderings $(R_{j_1},R_{j_2},...,R_{j_a})$ of autonomous individuals there exists an $R \in C$ such that $R = f\,(R_{j_1},...,R_{j_a}, (\forall i \in H)\ (R_i = R), (\forall i \in D)\ (R_i = \overline{R}\,))$, i.e., there exists an equilibrium iff f is a simple game.
Proof : Necessity

Let $S = \{x_1,...,x_s\}$. Suppose f is not a simple game. Then $(\exists x_k,x_l \in S)$ $(\exists (R_1|\{x_k,x_l\},...,R_n|\{x_k,x_l\}) \in (C|\{x_k,x_l\})^n)$ $[\{i \in N \mid x_k P_i x_l\} = V \notin W \wedge x_k P x_l]$. Denote by V' the set $\{i \in N \mid x_k I_i x_l\}$. By conditions I, N and M it follows that V is nonempty and is $(N - V')$-winning. Let V^o be the smallest subset or one of the smallest subsets of V' such that V is $(N - V^o)$-winning. $V \notin W$ entails $V^o \neq \emptyset$. Let individual $r \in V^o$. By the definition of V^o it follows that V is not $(N - (V^o - \{r\}))$-winning. Consider the decomposition (A,H,D) of N given by $(A = N - \{r\}, H = \emptyset, D = \{r\})$ and the following profile of orderings for autonomous individuals :
$(\forall i \in V)$ $(x_1P_ix_2P_i...P_ix_s)$
$(\forall i \in V^o - \{r\})$ $(x_1I_ix_2I_i...I_ix_s)$
$(\forall i \in N - (V \cup V^o))$ $(x_sP_ix_{s-1}P_i...P_ix_1)$
Now,
$R = f\,((\forall i \in V)\ (x_1P_i...P_ix_s), (\forall i \in V^o - \{r\})\ (x_1I_i...I_ix_s), (\forall i \in N - (V \cup V^o))\ (x_sP_i...P_ix_1), R_r = \overline{R}\,)$ with $R|\{x_1,x_2\} = x_1Px_2 \rightarrow x_2P_rx_1$
$\rightarrow$ V is $(N - (V^o - \{r\}))$-winning, contradicting the definition of V^o.
Therefore, we conclude :
$R|\{x_1,x_2\} = x_1Px_2 \rightarrow R \neq f\,((\forall i \in V)\ (x_1P_i...P_ix_s), (\forall i \in V^o - \{r\})\ (x_1I_i...I_ix_s), (\forall i \in N - (V \cup V^o))\ (x_sP_i...P_ix_1), R_r = \overline{R}\,)$. (1)
$R = f\,((\forall i \in V)\ (x_1P_i...P_ix_s), (\forall i \in V^o - \{r\})\ (x_1I_i...I_ix_s), (\forall i \in N - (V \cup V^o))\ (x_sP_i...P_ix_1), R_r = \overline{R}\,)$ with $R|\{x_1,x_2\} = x_2Px_1 \rightarrow x_1P_rx_2$
$\rightarrow$ $V \cup \{r\}$ is not $(N - (V^o - \{r\}))$-winning
$\rightarrow$ V is not $(N - V^o)$-winning, as a consequence of conditions I, N and M, contradicting the hypothesis.
Therefore,
$R|\{x_1,x_2\} = x_2Px_1 \rightarrow R \neq f\,((\forall i \in V)\ (x_1P_i...P_ix_s), (\forall i \in V^o - \{r\})\ (x_1I_i...I_ix_s), (\forall i \in N - (V \cup V^o))\ (x_sP_i...P_ix_1), R_r = \overline{R}\,)$. (2)
Finally,
$R = f\,((\forall i \in V)\ (x_1P_i...P_ix_s), (\forall i \in V^o - \{r\})\ (x_1I_i...I_ix_s), (\forall i \in N - (V \cup V^o))\ (x_sP_i...P_ix_1), R_r = \overline{R}\,)$ with $R|\{x_1,x_2\} = x_1Ix_2 \rightarrow x_1I_rx_2$
$\rightarrow$ V is not $(N - V^o)$-winning, contradicting the hypothesis.
Consequently,
$R|\{x_1,x_2\} = x_1Ix_2 \rightarrow R \neq f\,((\forall i \in V)\ (x_1P_i...P_ix_s), (\forall i \in V^o - \{r\})\ (x_1I_i...I_ix_s), (\forall i \in N - (V \cup V^o))\ (x_sP_i...P_ix_1), R_r = \overline{R}\,)$. (3)
As social binary weak preference relation is connected, (1)-(3) establish that there is no equilibrium for the situation under consideration.
Sufficiency

Suppose there exists a decomposition (A,H,D) of N, $A = \{j_1,j_2,...,j_a\}$, and a profile of autonomous individuals' orderings $(R^o_{j_1},R^o_{j_2},...,R^o_{j_a})$ such that $(\forall R \in C)$ $[R \neq f\,(R^o_{j_1},R^o_{j_2},...,R^o_{j_a}, (\forall i \in H)\ (R_i = R), (\forall i \in D)\ (R_i = \overline{R}\,))]$. For $x_k,x_l \in S$, $k \neq l$, designate by V_{kl}, V_{lk}, $V_{(kl)}$ the sets $\{i \in A \mid x_kP^o_ix_l\}$, $\{i \in A \mid x_lP^o_ix_k\}$, $\{i \in A \mid x_kI^o_ix_l\}$ respectively. Suppose $(\forall$ distinct $k,l \in \{1,...,s\})$ $[V_{kl} \cup H$ is $(N - V_{(kl)})$-winning $\vee$ $V_{lk} \cup H$ is $(N - V_{(kl)})$-winning $\vee$ (neither V_{kl} nor V_{lk} is $(N - (V_{(kl)} \cup H \cup D))$-winning)].
Consider $R^o \in C$ given by :
For $x_k,x_l \in S$, $k < l$,
$x_kP^ox_l$ if $V_{kl} \cup H$ is $(N - V_{(kl)})$-winning
$x_lP^ox_k$ if $V_{kl} \cup H$ is not $(N - V_{(kl)})$-winning and $V_{lk} \cup H$ is $(N - V_{(kl)})$-winning
$x_kI^ox_l$ if neither $V_{kl} \cup H$ nor $V_{lk} \cup H$ is $(N - V_{(kl)})$-winning.

Then, clearly, R^o = f ($R^o j_1$,$R^o j_2$,...,$R^o j_a$, (∀i ∈ H) (R_i = R^o), (∀i ∈ D) (R_i = $\overline{R^o}$)). This contradicts the hypothesis that (∀R ∈ C) [R ≠ f ($R^o j_1$,$R^o j_2$,...,$R^o j_a$, (∀i ∈ H) (R_i = R), (∀i ∈ D) (R_i = $\overline{R}$))]. Therefore we conclude (∃ distinct k,l ∈ {1,...,s}) [$V_{kl} \cup H$ is not (N − $V_{(kl)}$)-winning ∧ $V_{lk} \cup H$ is not (N − $V_{(kl)}$)-winning ∧ (V_{kl} is (N − ($V_{(kl)} \cup H \cup D$))-winning ∨ V_{lk} is (N − ($V_{(kl)} \cup H \cup D$))-winning)]. Neither $V_{kl} \cup H$ nor $V_{lk} \cup H$ being (N − $V_{(kl)}$)-winning implies that ($V_{kl} \notin W$ ∧ $V_{lk} \notin W$). From this we conclude, in view of the fact that [V_{kl} is (N − ($V_{(kl)} \cup H \cup D$))-winning ∨ V_{lk} is (N − ($V_{(kl)} \cup H \cup D$))-winning], that f is not a simple game. This establishes the theorem.

Theorem 2 : Let f : $T^n \mapsto$ C be a neutral and monotonic binary social decision rule. Then for every decomposition (A,H,D) of the set of individuals N, A = {j_1,j_2...,j_a}, and for every profile of orderings (Rj_1,Rj_2,...,Rj_a) for autonomous individuals there exists an R ∈ C such that R = f (Rj_1,...,Rj_a, (∀i ∈ H) (R_i = t(R)), (∀i ∈ D) (R_i = t($\overline{R}$))), i.e., there exists an equilibrium iff f is a simple game.
Proof: Necessity :

Let S = {x_1,...,x_s}. Suppose f is not a simple game. Then we can conclude, as in the necessity part of theorem 1, that there exist nonempty subsets of N, V and V′ such that V is (N − V′)-winning, V ∉ W and V ∩ V′ = ∅. Let V^o be the smallest subset or one of the smallest subsets of V′ such that V is (N − V^o)-winning. V ∉ W entails $V^o \neq \emptyset$. Let individual r ∈ V^o. By the definition of V^o it follows that V is not (N − (V^o − {r}))-winning. Consider the decomposition (A,H,D) of N given by (A = N − {r}, H = ∅, D = {r}) and the following profile of orderings for autonomous individuals :
(∀i ∈ V) ($x_1 P_i x_2 P_i ... P_i x_s$)
(∀i ∈ V^o − {r}) ($x_1 I_i x_2 I_i ... I_i x_s$)
(∀i ∈ N − (V ∪ V^o)) ($x_s P_i x_{s-1} P_i ... P_i x_1$).

Suppose R = f ((∀i ∈ V) ($x_1 P_i ... P_i x_s$), (∀i ∈ V^o − {r}) ($x_1 I_i ... I_i x_s$), (∀i ∈ N − (V ∪ V^o)) ($x_s P_i ... P_i x_1$), R_r = t($\overline{R}$)). Let x_k,x_l ∈ S, k < l. x_kP(t(R))x_l implies x_lP(t($\overline{R}$))x_k, and consequently $x_l P_r x_k$. As R is connected, x_kP(t(R))x_l implies x_kPx_l. Thus it follows that V is (N − (V^o − {r}))-winning, which contradicts the definition of V^o. Therefore, we conclude ∼ x_kP(t(R))x_l. Next suppose x_lP(t(R))x_k. This entails $x_k P_r x_l$; and x_lPx_k in view of connectedness of R. x_lPx_k implies that V ∪ {r} is not (N − (V^o − {r}))-winning. However, V being (N − V^o)-winning implies, by conditions I, N and M, that V ∪ {r} is (N − (V^o − {r}))-winning. This contradiction establishes that ∼ x_lP(t(R))x_k. Therefore, it follows that t(R) is given by : (∀k,l ∈ {1,2,...,s}) (x_kI(t(R))x_l). This entails that we must have (∀k,l ∈ {1,2,...,s}) [k < l → x_kPx_l], as V is (N − V^o)-winning. Thus R = x_1Px_2P...Px_s. But then t(R) = R, which contradicts (∀k,l ∈ {1,2,...,s}) (x_kI(t(R))x_l). This establishes that ∼ (∃R ∈ C) [R = f ((∀i ∈ V) ($x_1 P_i ... P_i x_s$), (∀i ∈ V^o − {r}) ($x_1 I_i ... I_i x_s$), (∀i ∈ N − (V ∪ V^o)) ($x_s P_i ... P_i x_1$), R_r = t($\overline{R}$)], i.e., the situation under consideration has no equilibrium.
Sufficiency :

Let f be a simple game. Consider any decomposition (A,H,D) of N, A = {j_1,...,j_a}, and any profile of orderings (Rj_1,Rj_2,...,Rj_a) for autonomous individuals. For x_k,x_l ∈ S, k ≠ l, designate by V_{kl}, V_{lk}, $V_{(kl)}$ the sets {i ∈ A | $x_k P_i$ x_l}, {i ∈ A | $x_l P_i$ x_k}, {i ∈ A | $x_k I_i$ x_l} respectively. Consider R^* ∈ C defined by :
For x_k,x_l ∈ S, k ≠ l,
$x_k P^* x_l$ iff V_{kl} ∈ W
$x_l P^* x_k$ iff V_{lk} ∈ W
$x_k I^* x_l$ iff V_{kl} ∉ W and V_{lk} ∉ W.

If $x_k P^* x_l$ then V_{kl} ∈ W. Consequently f yields x_kPx_l corresponding to (Rj_1,Rj_2,...,Rj_a, (∀i ∈ H) (R_i = t(R^*)), (∀i ∈ D) (R_i = t($\overline{R^*}$))) irrespective of whether x_kP(t(R^*))x_l or x_kI(t(R^*))x_l holds. Similarly $x_l P^* x_k$ implies V_{lk} ∈ W, which in turn implies that f must yield x_lPx_k corresponding to (Rj_1,...,Rj_a, (∀i ∈ H) (R_i = t(R^*)), (∀i ∈ D) (R_i = t($\overline{R^*}$))) irrespective of whether x_lP(t(R^*))x_k or x_lI(t(R^*))x_k holds. $x_k I^* x_l$ implies x_kI(t(R^*))x_l, consequently corresponding to the profile of orderings (Rj_1,...,Rj_a, (∀i ∈ H) (R_i = t(R^*)), (∀i ∈ D) (R_i = t($\overline{R^*}$))) we have {i ∈ N | $x_k P_i x_l$} = V_{kl} and {i ∈ N | $x_l P_i x_k$} = V_{lk}. As $x_k I^* x_l$ implies that V_{kl} ∉ W and V_{lk} ∉ W, and f is a simple game it follows that f yields x_kIx_l corresponding to (Rj_1,...,Rj_a, (∀i ∈ H) (R_i = t(R^*)), (∀i ∈ D) (R_i = t($\overline{R^*}$))). This proves that R^* is an equilibrium for the situation under consideration, i.e., R^* = f (Rj_1,...,Rj_a, (∀i ∈ H) (R_i = t(R^*)), (∀i ∈ D) (R_i = t($\overline{R^*}$))). This establishes the theorem.

Theorem 3 : Let f be a neutral and monotonic binary social decision rule. Then, there exist a decomposition (A,H,D) of the set of individuals N, A = {j_1,j_2...,j_a}, a profile of orderings (Rj_1,Rj_2,...,Rj_a) for autonomous individuals, and a linear ordering L of the set of social alternatives S such that L = f (Rj_1,...,Rj_a, (∀i ∈ H) (R_i = L), (∀i ∈ D) (R_i = $\overline{L}$)) and $\overline{L}$ = f (R_{j_1},...,R_{j_a}, (∀i ∈ H) (R_i = $\overline{L}$), (∀i ∈ D) (R_i = L)) iff f is non-null.

Proof : Suppose f is non-null. Then, by conditions I, N and M it follows that N $\in$ W. N $\in$ W implies that $\exists$V $\subseteq$ N such that V $\in W_m$. Now consider the decomposition (A,H,D) of N given by (A = N $-$ V, H = V, D = $\emptyset$), and the following profile of orderings for autonomous individuals : ($\forall$i $\in$ A) ($\forall$x,y $\in$ S) (xI_iy). Let L be a linear ordering of S. We obtain, [f (($\forall$i $\in$ A) ($\forall$x,y $\in$ S) (xI_iy), ($\forall$i $\in$ V) (R_i = L)) = L and f (($\forall$i $\in$ A) ($\forall$x,y $\in$ S) (xI_iy), ($\forall$i $\in$ V) (R_i = $\overline{L}$)) = $\overline{L}$], as a consequence of V being winning, which establishes that both L and $\overline{L}$ are equilibria.

The other part of the theorem being trivial, this completes the proof.

3. Characterization Theorems with Single Non-Autonomous Individual

Theorem 4 : Let f : $C^n \mapsto C$ be a neutral and monotonic binary social decision rule. Then, for every decomposition (A,D) of the set of individuals N with #D = 1, A = {j_1,j_2,...,j_{n-1}}, and for every profile of orderings (R_{j_1},R_{j_2},...,$R_{j_{n-1}}$) of autonomous individuals there exists an R $\in$ C such that R = f (R_{j_1},R_{j_2},...,$R_{j_{n-1}}$, ($\forall$i $\in$ D) (R_i = $\overline{R}$)), i.e., there exists an equilibrium iff f is a simple game.
Proof : Follows from the proof of theorem 1. Necessity follows as the decomposition of N considered there was (A = N $-$ {r}, D = {r}). In the proof of the sufficiency part, it was shown that if f is a simple game, then for every decomposition (A,H,D) of N, A = {j_1,...,j_a}, and for every profile of orderings (R_{j_1},...,R_{j_a}) of autonomous individuals there is an equilibrium. Consequently it follows that if f is a simple game then for every decomposition (A,D) of N with #D = 1, A = {j_1,...,j_{n-1}}, and for every profile of orderings (R_{j_1},...,$R_{j_{n-1}}$) of autonomous individuals there is an equilibrium.

Theorem 5 : Let f : $T^n \mapsto C$ be a neutral and monotonic binary social decision rule. Then, for every decomposition (A,D) of the set of individuals N with #D = 1, A = {j_1,j_2,...,j_{n-1}}, and for every profile of orderings (R_{j_1},R_{j_2},...,$R_{j_{n-1}}$) for autonomous individuals there exists an R $\in$ C such that R = (R_{j_1},R_{j_2},...,$R_{j_{n-1}}$, ($\forall$i $\in$ D) (R_i = t($\overline{R}$))), i.e., there exists an equilibrium iff f is a simple game.
Proof : Follows from the proof of theorem 2 for the same reasons as theorem 4 follows from the proof of theorem 1.

Theorem 6 : Let f be a neutral and monotonic binary social decision rule. Then, there exist a decomposition (A,H) of the set of individuals N with #H = 1, A = {j_1,j_2,...,j_{n-1}}, a profile of orderings (R_{j_1},R_{j_2},...,$R_{j_{n-1}}$) for autonomous individuals, and a linear ordering L of the set of social alternatives such that L = f (R_{j_1},R_{j_2},...,$R_{j_{n-1}}$, R_i = L for i $\in$ H) and $\overline{L}$ = f (R_{j_1},R_{j_2},...,$R_{j_{n-1}}$, R_i = $\overline{L}$ for i $\in$ H), i.e., both L and $\overline{L}$ are equilibria iff f is such that for some sub-society (N $-$ M), M $\subset$ N, either there is a minimal (N $-$ M)-winning coalition consisting of a single individual or there exist distinct minimal (N $-$ M)-winning coalitions V_1 and V_2 such that $V_1 \cap V_2$ is a singleton.
Proof : <u>Sufficiency</u>

First suppose that for some sub-society (N $-$ M), M $\subset$ N, there exists a minimal (N $-$ M)-winning coalition consisting of a single individual, say, individual k. Consider the decomposition (A,H) of N given by : (A = N $-$ {k}, H = {k}), and any profile of autonomous individuals' orderings (R_1,...,R_{k-1},R_{k+1},...,R_n) such that ($\forall$i $\in$ M) ($\forall$x,y $\in$ S) (xI_iy). Let L be a linear ordering of S. Then, [f (R_1,...,R_{k-1}, R_k = L, R_{k+1},...,R_n) = L $\wedge$ f (R_1,...,R_{k-1}, R_k = $\overline{L}$, R_{k+1},...,R_n) = $\overline{L}$].

Next suppose that for some sub-society (N $-$ M), M $\subset$ N, there exist distinct minimal (N $-$ M)-winning coalitions V_1 and V_2 such that $V_1 \cap V_2$ is a singleton. Let $V_1 \cap V_2$ = {k}. Consider the decomposition (A,H) of N given by : (A = N $-$ {k}, H = {k}), and the following profile of orderings for autonomous individuals :
($\forall$i $\in$ N $-$ ($V_1 \cup V_2$)) ($\forall$x,y $\in$ S) (xI_iy)
($\forall$i $\in$ V_1 $-$ {k}) (R_i = L)
($\forall$i $\in$ V_2 $-$ {k}) (R_i = $\overline{L}$),
where L is a linear ordering of S.
As V_1 and V_2 are (N $-$ M)-winning, we obtain : [f (($\forall$i $\in$ N $-$ ($V_1 \cup V_2$)) ($\forall$x,y $\in$ S) (xI_iy), ($\forall$i $\in$ V_1 $-$ {k}) (R_i = L), ($\forall$i $\in$ V_2 $-$ {k}) (R_i = $\overline{L}$), R_k = L) = L $\wedge$ f (($\forall$i $\in$ N $-$ ($V_1 \cup V_2$)) ($\forall$x,y $\in$ S) (xI_iy), ($\forall$i $\in$ V_1 $-$ {k}) (R_i = L), ($\forall$i $\in$ V_2 $-$ {k}) (R_i = $\overline{L}$), R_k = $\overline{L}$) = $\overline{L}$].
<u>Necessity</u>

Suppose there exist a decomposition (A,H) of N with #H = 1, H = {k}, A = {1,2,...,k-1,k+1,...,n}, a profile of autonomous individuals' orderings (R_1,R_2,...,R_{k-1},R_{k+1},...,R_n), and a linear ordering L of the set of social alternatives S such that L = f (R_1,R_2,...,R_{k-1}, R_k = L, R_{k+1},...,R_n) and $\overline{L}$ = f (R_1,R_2,...,R_{k-1}, R_k = $\overline{L}$,

R_{k+1},...,R_n). Suppose xLy and y$\overline{L}$ x, x,y $\in$ S, x $\neq$ y. Denote {i $\in$ A $\mid$ xP_iy}, {i $\in$ A $\mid$ yP_ix}, {i $\in$ A $\mid$ xI_iy} by V_{xy}, V_{yx}, $V_{(xy)}$ respectively.
L = f (R_1,...,R_{k-1}, R_k = L, R_{k+1},...,R_n) $\rightarrow$ $V_{xy} \cup$ {k} is (N $-$ $V_{(xy)}$)-winning, as a consequence of conditions I, M and N.
$\rightarrow$ $\exists$ $V_1 \subseteq$ V_{xy} $\cup$ {k} such that V_1 is minimally (N $-$ $V_{(xy)}$)-winning.
$\overline{L}$ = f (R_1,...,R_{k-1}, R_k = $\overline{L}$, R_{k+1},...,R_n) $\rightarrow$ $V_{yx} \cup$ {k} is (N $-$ $V_{(xy)}$)-winning, as a consequence of conditions I, M and N.
$\rightarrow$ $\exists$ $V_2 \subseteq$ V_{yx} $\cup$ {k} such that V_2 is minimally (N $-$ $V_{(xy)}$)-winning.
As $V_1 \cap V_2$ $\neq$ $\emptyset$, we conclude {k} = $V_1 \cap V_2$. It follows that either there exist distinct minimal (N $-$ $V_{(xy)}$)-winning coalitions V_1 and V_2 such that $V_1 \cap V_2$ is a singleton or there exists an individual who is minimally (N $-$ $V_{(xy)}$)-winning. This establishes the theorem.

4. The Method of Majority Decision

Theorem 7 : Let f : C^n $\mapsto$ C be the method of majority decision. Let (A,H,D) be a decomposition of the set of individuals N, A = {j_1,j_2...,j_a}, and (Rj_1,Rj_2,...,Rj_a) a profile of orderings of autonomous individuals. Then, ($\forall$R $\in$ C) [R $\neq$ f (Rj_1,...,Rj_a, ($\forall$i $\in$ H) (R_i = R), ($\forall$i $\in$ D) (R_i = $\overline{R}$)], i.e., there is no equilibrium iff ($\exists$x,y $\in$ S) [d - h $\geq$ $\mid$#{i $\in$ A $\mid$ xP_iy} - #{i $\in$ A $\mid$ yP_ix}$\mid$ $>$ 0].
Proof : <u>Sufficiency</u>

Denote #{i $\in$ A $\mid$ xP_iy} and #{i $\in$ A $\mid$ yP_ix} by a_{xy} and a_{yx} respectively. Suppose d - h $\geq$ a_{xy} - a_{yx} $>$ 0, x,y $\in$ S.
d + a_{yx} $\geq$ h + a_{xy} $\rightarrow$ [R|{x,y} = xPy $\rightarrow$ R $\neq$ f (Rj_1,...,Rj_a, ($\forall$i $\in$ H) (R_i = R),
($\forall$i $\in$ D) (R_i = $\overline{R}$))] (i)
(d + a_{yx} $\geq$ h + a_{xy}) $\wedge$ (a_{xy} $>$ a_{yx}) $\rightarrow$ d + a_{xy} $>$ h + a_{yx}
d + a_{xy} $>$ h + a_{yx} $\rightarrow$ [R|{x,y} = yPx $\rightarrow$ R $\neq$ f (Rj_1,...,Rj_a, ($\forall$i $\in$ H) (R_i = R),
($\forall$i $\in$ D) (R_i = $\overline{R}$))] (ii)
a_{xy} $>$ a_{yx} $\rightarrow$ [R|{x,y} = xIy $\rightarrow$ R $\neq$ f (Rj_1,...,Rj_a, ($\forall$i $\in$ H) (R_i = R),
($\forall$i $\in$ D) (R_i = $\overline{R}$))] (iii)
(i)-(iii) establish that ($\forall$R $\in$ C) [R $\neq$ f (Rj_1,...,Rj_a, ($\forall$i $\in$ H) (R_i = R), ($\forall$i $\in$ D) (R_i = $\overline{R}$))], i.e., there is no equilibrium.
<u>Necessity</u>

Suppose ($\forall$x,y $\in$ S) [d - h $<$ $\mid a_{xy}$ - $a_{yx}\mid$ $\vee$ a_{xy} = a_{yx}]. Let R^o $\in$ C be defined by : ($\forall$x,y $\in$ S) [xR^oy $\leftrightarrow$ a_{xy} $\geq$ a_{yx}]. Now,
a_{xy} $>$ a_{yx} $\rightarrow$ a_{xy} + h $>$ a_{yx} + d
$\rightarrow$ f yields xPy corresponding to (Rj_1,...,Rj_a, ($\forall$i $\in$ H) (R_i = R^o),
($\forall$i $\in$ D) (R_i = $\overline{R^o}$)) (1)
a_{yx} $>$ a_{xy} $\rightarrow$ a_{yx} + h $>$ a_{xy} + d
$\rightarrow$ f yields yPx corresponding to (Rj_1,...,Rj_a, ($\forall$i $\in$ H) (R_i = R^o),
($\forall$i $\in$ D) (R_i = $\overline{R^o}$)) (2)
a_{xy} = a_{yx} $\rightarrow$ f yields xIy corresponding to (Rj_1,...,Rj_a, ($\forall$i $\in$ H) (R_i = R^o),
($\forall$i $\in$ D) (R_i = $\overline{R^o}$)) (3)
(1) - (3) establish that R^o = f (Rj_1,...,Rj_a, ($\forall$i $\in$ H) (R_i = R^o), ($\forall$i $\in$ D) (R_i = $\overline{R^o}$)), i.e., R^o is an equilibrium. This establishes the theorem.

Theorem 8 : Let f : T^n $\mapsto$ C be the method of majority decision. Let (A,H,D) be a decomposition of the set of individuals N, A = {j_1,j_2...,j_a}, and (Rj_1,Rj_2,...,Rj_a) a profile of orderings of autonomous individuals. Then ($\forall$R $\in$ C) [R $\neq$ f (Rj_1,...,Rj_a, ($\forall$i $\in$ H) (R_i = t(R)), ($\forall$i $\in$ D) (R_i = t($\overline{R}$)))], i.e., there is no equilibrium iff ($\exists$x,y $\in$ S) [d -h $\geq$ $\mid$#{i $\in$ A $\mid$ xP_iy} - #{i $\in$ A $\mid$ yP_ix}$\mid$ $>$ 0 $\wedge$ R^*|{x,y} = t(R^*)|{x,y}], where R^* = f(Rj_1,...,Rj_a, ($\forall$i $\in$ H $\cup$ D) ($\forall$z,w $\in$ S) (zI_iw)).
Proof : <u>Sufficiency</u>

Let x,y $\in$ S. Denote #{i $\in$ A $\mid$ xP_iy} and #{i $\in$ A $\mid$ yP_ix} by a_{xy} and a_{yx} respectively. Suppose [d - h $\geq$ a_{xy} - a_{yx} $>$ 0 $\wedge$ R^*|{x,y} = t(R^*)|{x,y}].
d - h $\geq$ a_{xy} - a_{yx} $\rightarrow$ d + a_{yx} $\geq$ h + a_{xy} (1)
(1) $\wedge$ a_{xy} - a_{yx} $>$ 0 $\rightarrow$ d + a_{xy} $>$ h + a_{yx}. (2)
Suppose there is an R $\in$ C such that : R = f (Rj_1,...,Rj_a, ($\forall$i $\in$ H) (R_i = t(R)),

$(\forall i \in D)\ (R_i = t(\overline{R}\ )))$.

$xP(t(R))y \rightarrow xPy$, as R is connected (3)

$xP(t(R))y \wedge (1) \rightarrow \#\{i \in N \mid yP_ix\} = d + a_{yx} \geq h + a_{xy} = \#\{i \in N \mid xP_iy\}$ (4)

As (3) and (4) contradict each other, we conclude that :

$\sim xP(t(R))y$ (5)

$yP(t(R))x \rightarrow yPx$, by connectedness of R (6)

$yP(t(R))x \wedge (2) \rightarrow \#\{i \in N \mid xP_iy\} = d + a_{xy} > h + a_{yx} = \#\{i \in N \mid yP_ix\}$ (7)

As (6) and (7) contradict each other, it follows that :

$\sim yP(t(R))x$ (8)

$xI(t(R))y \rightarrow \#\{i \in N \mid xP_iy\} = a_{xy} > a_{yx} = \#\{i \in N \mid yP_ix\}$ (9)

$(9) \rightarrow \sim (xI(t(R))y \wedge xIy) \wedge \sim (xI(t(R))y \wedge yPx)$ (10)

(5), (8) and (10) imply that it must be the case that xI(t(R))y and xPy hold.

$xI(t(R))y \rightarrow xt(R)y \wedge yt(R)x$

$yt(R)x \rightarrow yR^kx$, for some positive integer k

$\rightarrow (\exists z_1,z_2,...,z_{k-1} \in S)\ (yRz_1 \wedge z_1Rz_2 \wedge ... \wedge z_{k-1}Rx)$

$xRy \wedge (yRz_1 \wedge z_1Rz_2 \wedge ... \wedge z_{k-1}Rx) \rightarrow xt(R)y \wedge (yt(R)z_1 \wedge z_1t(R)z_2 \wedge ... \wedge z_{k-1}t(R)x)$

$\rightarrow yI(t(R))z_1 \wedge z_1I(t(R))z_2 \wedge ... \wedge z_{k-1}I(t(R))x$, by transitivity of t(R).

$(xPy \wedge yRz_1 \wedge z_1Rz_2 \wedge ... \wedge z_{k-1}Rx) \wedge (xI(t(R))y \wedge yI(t(R))z_1 \wedge z_1I(t(R))z_2 \wedge ... \wedge z_{k-1}I(t(R))x) \rightarrow xP^*y \wedge yR^*z_1 \wedge z_1R^*z_2 \wedge ... \wedge z_{k-1}R^*x$, by the definition of R^* and condition I

$\rightarrow xI(t(R^*))y \wedge xP^*y$.

Thus we have shown that : $xI(t(R))y \wedge xPy \rightarrow R^*|\{x,y\} \neq t(R^*)|\{x,y\}$.

Consequently we conclude that :

$\sim (xI(t(R))y \wedge xPy)$ (11)

(5), (8), (10) and (11) imply that our supposition that there is an $R \in C$ such that $R = f\ (Rj_1,...,Rj_a, (\forall i \in H)\ (R_i = t(R)), (\forall i \in D)\ (R_i = t(\overline{R}\ )))$ is false. Therefore, if $(\exists x,y \in S)\ [d - h \geq |a_{xy} - a_{yx}| > 0 \wedge R^*|\{x,y\} = t(R^*)|\{x,y\}]$, then there is no equilibrium.

<u>Necessity</u>

Suppose $\sim (\exists x,y \in S)\ [d - h \geq |a_{xy} - a_{yx}| > 0 \wedge R^*|\{x,y\} = t(R^*)|\{x,y\}]$, i.e., $(\forall x,y \in S)\ [d - h < |a_{xy} - a_{yx}| \vee a_{xy} = a_{yx} \vee R^*|\{x,y\} \neq t(R^*)|\{x,y\}]$ (i)

Consider any $x,y \in S$.

$xP(t(R^*))y \rightarrow xP^*y$, by connectedness of R^*

$\rightarrow a_{xy} > a_{yx}$, by the definition of R^*

$\rightarrow d - h < a_{xy} - a_{yx}$, by (i)

$\rightarrow d + a_{yx} < h + a_{xy}$

$\rightarrow [f\ (Rj_1,...,Rj_a, (\forall i \in H)\ (R_i = t(R^*)), (\forall i \in D)\ (R_i = t(\overline{R^*})))]|\{x,y\} = R^*|\{x,y\}$ (ii)

$yP(t(R^*))x \rightarrow yP^*x$, as R^* is connected

$\rightarrow a_{yx} > a_{xy}$, by the definition of R^*

$\rightarrow d - h < a_{yx} - a_{xy}$, by (i)

$\rightarrow d + a_{xy} < h + a_{yx}$

$\rightarrow [f\ (Rj_1,...,Rj_a, (\forall i \in H)\ (R_i = t(R^*)), (\forall i \in D)\ (R_i = t(\overline{R^*})))]|\{x,y\} = R^*|\{x,y\}$ (iii)

$xI(t(R^*))y \rightarrow [f\ (Rj_1,...,Rj_a, (\forall i \in H)\ (R_i = t(R^*)), (\forall i \in D)\ (R_i = t(\overline{R^*})))]|\{x,y\} = R^*|\{x,y\}$, by the definition of R^* and condition I. (iv)

From (ii) - (iv) it follows, in view of connectedness of $t(R^*)$, that $[f\ (Rj_1,...,Rj_a, (\forall i \in H)\ (R_i = t(R^*)), (\forall i \in D)\ (R_i = t(\overline{R^*})))]|\{x,y\} = R^*|\{x,y\}$. This proves that $R^* = f(Rj_1,...,Rj_a, (\forall i \in H)\ (R_i = t(R^*)), (\forall i \in D)\ (R_i = t(\overline{R^*})))$, i.e., R^* is an equilibrium. This establishes the theorem.

Theorem 9 : Let f be the method of majority decision. Let (A,H,D) be a decomposition of the set of individuals N, $A = \{j_1,j_2,...,j_a\}$, and $(Rj_1,Rj_2,...,Rj_a)$ a profile of orderings of autonomous individuals. Then, there exists a linear ordering L of the set of alternatives such that $L = f\ (Rj_1,...,Rj_a, (\forall i \in H)\ (R_i = L), (\forall i \in D)\ (R_i = \overline{L}\ ))$ and $\overline{L} = f\ (Rj_1...,Rj_a, (\forall i \in H)\ (R_i = \overline{L}\ ), (\forall i \in D)\ (R_i = L))$, i.e., both L and $\overline{L}$ are equilibria iff $(\forall x,y \in S)\ [h - d > |\#\{i \in A \mid xP_iy\} - \#\{i \in A \mid yP_ix\}|\ ]$.

Proof : <u>Sufficiency</u>

Denote $\#\{i \in A \mid xP_iy\}$ and $\#\{i \in A \mid yP_ix\}$ by a_{xy} and a_{yx} respectively.

$h - d > |a_{xy} - a_{yx}| \rightarrow (h - d > a_{xy} - a_{yx}) \wedge (h - d > a_{yx} - a_{xy})$

$\rightarrow (h + a_{yx} > d + a_{xy}) \wedge (h + a_{xy} > d + a_{yx})$

Consequently,

$(\forall x,y \in S)$ $[h - d > |a_{xy} - a_{yx}|] \rightarrow L = f(Rj_1,...,Rj_a, (\forall i \in H)(R_i = L), (\forall i \in D)(R_i = \overline{L})) \wedge \overline{L} = f(Rj_1...,Rj_a, (\forall i \in H)(R_i = \overline{L}), (\forall i \in D)(R_i = L))$, where L is any linear ordering of S.

<u>Necessity</u>

Suppose $(\exists x,y \in S)$ $[h - d \leq |a_{xy} - a_{yx}|]$.

$h - d \leq |a_{xy} - a_{yx}| \rightarrow [h - d \leq a_{xy} - a_{yx} \vee h - d \leq a_{yx} - a_{xy}]$

$\rightarrow [h + a_{yx} \leq d + a_{xy} \vee h + a_{xy} \leq d + a_{yx}]$

$h + a_{yx} \leq d + a_{xy} \rightarrow$ [L is a linear ordering of S and $L|\{x,y\} = yPx \rightarrow L \neq f(Rj_1,...,Rj_a, (\forall i \in H)(R_i = L), (\forall i \in D)(R_i = \overline{L}))$]

$h + a_{xy} \leq d + a_{yx} \rightarrow$ [L is a linear ordering of S and $L|\{x,y\} = xPy \rightarrow L \neq f(Rj_1,...,Rj_a, (\forall i \in H)(R_i = L), (\forall i \in D)(R_i = \overline{L}))$].

Thus for any linear ordering L of S we have : $[L \neq f(Rj_1,...,Rj_a, (\forall i \in H)(R_i = L), (\forall i \in D)(R_i = \overline{L})) \vee \overline{L} \neq f(Rj_1,...,Rj_a, (\forall i \in H)(R_i = \overline{L}), (\forall i \in D)(R_i = L))]$, i.e., there is no linear ordering L of S such that both L and $\overline{L}$ are equilibria. This establishes the theorem.

Remark 4 : Let f be the method of majority decision. Let (A,H,D) be a decomposition of N, $A = \{j_1,...,j_a\}$, and $(Rj_1,...,Rj_a)$ a profile of autonomous individuals' orderings. Suppose S = {x,y}. Then, from theorems 7, 8 and 9, it follows that :

(i) there is no equilibrium iff $d - h \geq |\#\{i \in A \mid xP_iy\} - \#\{i \in A \mid yP_ix\}| > 0$

and

(ii) both xPy and yPx are equilibria iff $h - d > |\#\{i \in A \mid xP_iy\} - \#\{i \in A \mid yP_ix\}|$.

Remark 5 : From theorems 7, 8 and 9 we conclude : If d = h then under the method of majority decision (i) for every profile of autonomous individuals' orderings there is an equilibrium and (ii) there do not exist a profile of autonomous individuals' orderings and a linear ordering L of S such that both L and $\overline{L}$ are equilibria.

Remark 6 : Let (A,H,D) be any given decomposition of N. Then, from theorems 7, 8 and 9 we conclude that, under the method of majority decision both (i) and (ii) cannot be true.

(i) There exists a profile of autonomous individuals' orderings corresponding to which there is no equilibrium.

(ii) There exist a profile of autonomous individuals' orderings and a linear ordering L of S such that both L and $\overline{L}$ are equilibria.

Properties of the method of majority decision mentioned in remarks 5 and 6 do not hold in general. The following examples show that these properties do not hold in general even if one restricts attention to the subclass of binary social decision rules satisfying the conditions of neutrality, monotonicity and anonymity.

Example 1 :

Let S = {x,y}, N = {1,...,n}, n = 2k + 3, $k \geq 2$, a = 2k, h = 2, d = 1. Let the social decision rule be defined by :

$(\forall z,w \in S)$ $[zRw \leftrightarrow \sim [\#\{i \in N \mid wP_iz\} \geq k + 2 \vee [(\forall i \in N)(wR_iz) \wedge (\exists i \in N)(wP_iz)]]]$.

Denote $\#\{i \in A \mid xP_iy\}$, $\#\{i \in A \mid yP_ix\}$, $\#\{i \in A \mid xI_iy\}$, $\#\{i \in N \mid xP_iy\}$, $\#\{i \in N \mid yP_ix\}$ and $\#\{i \in N \mid xI_iy\}$ by a_{xy}, a_{yx}, $a_{(xy)}$, n_{xy}, n_{yx} and $n_{(xy)}$ respectively.

(i) For any profile of autonomous individuals' orderings such that $a_{xy} = a_{yx} = k$ both xPy and yPx are equilibria as $a_{xy} + h = a_{yx} + h = k + 2$.

(ii) For any profile of autonomous individuals' orderings such that $a_{xy} = k - 1$ and $a_{(xy)} = k + 1$ there is no equilibrium as,

$a_{xy} + h = k + 1$ and $d = 1 \rightarrow$ xPy is not an equilibrium

$a_{xy} = k - 1$ and $a_{yx} = 0 \rightarrow$ xIy is not an equilibrium

$h = 2$ and $a_{xy} + d = k \rightarrow$ yPx is not an equilibrium.

Example 2 :

Let S = {x,y}, N = {1,...,n}, n = 3k + 3, $k \geq 1$, a = 3k + 1, h = 1, d = 1. Let the social decision rule be defined by :

$(\forall z,w \in S)$ $[zRw \leftrightarrow \sim [\#\{i \in N \mid wP_iz\} > \frac{2}{3}[\#\{i \in N \mid wP_iz\} + \#\{i \in N \mid zP_iw\}]]]$.

We use the same notation as in example 1.

For any profile of autonomous individuals' orderings such that $a_{xy} = 2k + 1$ and $a_{yx} = k$ there is no equilibrium as,

$a_{xy} + h = 2k + 2$ and $a_{yx} + d = k + 1 \rightarrow$ xPy is not an equilibrium

$a_{xy} = 2k + 1$ and $a_{yx} = k \rightarrow$ xIy is not an equilibrium

$a_{yx} + h = k + 1$ and $a_{xy} + d = 2k + 2 \rightarrow$ yPx is not an equilibrium.

The following result, however, holds for the class of social decision rules satisfying conditions I, N, M and A.

Theorem 10 : Let f be a neutral, monotonic and anonymous binary social decision rule. Let (A,H,D) be a decomposition of N such that $d \geq h$, $A = \{j_1,...,j_a\}$. Then, there do not exist a profile of autonomous individuals' orderings $(R_{j_1},...,R_{j_a})$ and a linear ordering L of S such that $L = f(R_{j_1},...,R_{j_a}, (\forall i \in H)(R_i = L), (\forall i \in D)(R_i = \overline{L}))$ and $\overline{L} = f(R_{j_1},...,R_{j_a}, (\forall i \in H)(R_i = \overline{L}), (\forall i \in D)(R_i = L))$, i.e., both L and $\overline{L}$ are equilibria.

Proof : Suppose there exist a profile of autonomous individuals' orderings $(R_{j_1},...,R_{j_a})$ and a linear ordering L of S such that both L and $\overline{L}$ are equilibria. Let $x,y \in S$, $x \neq y$. Assume xLy and $y\overline{L}x$. Denote $\#\{i \in A \mid xP_iy\}$ and $\#\{i \in A \mid yP_ix\}$ by a_{xy} and a_{yx} respectively. As f satisfies conditions I, N, M and A it follows that :

$L = f(R_{j_1},...,R_{j_a}, (\forall i \in H)(R_i = L), (\forall i \in D)(R_i = \overline{L})) \rightarrow a_{xy} + h > a_{yx} + d$ (1)

$\overline{L} = f(R_{j_1},...,R_{j_a}, (\forall i \in H)(R_i = \overline{L}), (\forall i \in D)(R_i = L)) \rightarrow a_{yx} + h > a_{xy} + d$ (2)

$(1) \wedge (2) \rightarrow h > d$

As $h > d$ contradicts the hypothesis $d \geq h$, the theorem is established.

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