Discussion Paper Series

STABILITY AND TRANSITIVITY

by

Satish K. Jain

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Introduction

In this paper we will show that transitivity is a necessary and sufficient condition for stability for the class of binary social decision rules which satisfy the Pareto-criterion. However if we consider the class of all binary social decision rules then transitivity does not turn out to be a necessary condition for stability. In fact, transitivity is not a necessary condition for stability even for the class of binary social decision rules which satisfy the weak Pareto - criterion as can be seen by the following example.

Example 1: Let $S = \{x,y,z\}$ be the set of social alternatives and let the social decision rule f be characterized as follows:

(a) For all ordered pairs of alternatives (s,t), if individual 1 prefers s to t the society does likewise.

(b) For all pairs of alternatives $(s,t) \neq (x,y)$, if individual 1 is indifferent between s and t then society is also indifferent between s and t.

(c) If individual 1 is indifferent between x and y then the society prefers x to y.

It is obvious that the above social decision rule is immune from strategic manipulation and hence is stable. However this social decision rule does not always yield transitive social preference relation. If individual 1 is indifferent among all alternatives then the social preference relation yielded by f violates transitivity.

In [3] Pattanaik has shown that for the class of binary social decision rules which satisfy, the Pareto - criterion and are neutral, weak resoluteness [1] is a necessary condition for stability. As every binary social decision rule which satisfies the Pareto - criterion and always yields transitive social preference relation is weakly resolute, it follows that weak resoluteness is a necessary condition for stability for the class of binary social decision rules which satisfy the Pareto-criterion. So a hyproduct of our result is a generalization of Pattanaik's theorem. Weak resoluteness is a necessary condition for stability for all Pareto - inclusive binary social decision rules irrespective of whether they satisfy neutrality or not.

The paper is divided into two sections. In the first section we present the necessary definitions and assumptions. In the second section we prove the theorem that transitivity is a necessary and sufficient condition for stability for the class of Pareto - inclusive binary functions.

<u>Definitions and Assumptions</u>

The set of social alternatives (S) will be assumed to be finite and the number of elements (n) in S will be assumed to be at least 3. Alternatives are defined in such a way that they are mutually exclusive. The set of individuals will be denoted by N. The number of individuals (L) will be assumed to be greater than one and finite. Every individual $i \in N$ will be assumed to have an ordering R_i defined over S.

For every binary preference relation R, we define the strict preference relation P and the indifference relation I in the usual way, i.e., xPy iff xRy and not yRx; and xIy iff xRy and yRx.

Definition 1: A social decision rule (SDR) is a functional relation f such that for any set of L individual orderings $R_1, \cdots, R_L$ (one ordering for each individual), one and only one reflexive and connected social preference relation R is determined,

$$R = f(R_1, \cdots, R_L).$$

Definition 2: Condition of Unrestricted Domain (U): The domain of the f must include all logically possible combinations of individual orderings.

Definition 3: Condition of Independence of Irrelevant Alternatives (I): Let R and R' be the social binary relations determined by f corresponding respectively to two sets of individual preferences, $(R_1, \cdots, R_L)$ and $(R'_1, \cdots, R'_L)$. If for all pairs of alternatives x,y, in a subset A of S; $xR_iy \longleftrightarrow xR'_iy$, for all i, then $xRy \longleftrightarrow xR'y$, for all $x,y \in A$.

A function (SDR) which satisfies the condition of independence of irrelevant alternatives will be called a binary function. We now define the conditions of monotonicity and weak monotonicity for social decision rules which satisfy condition I.

Definition 4: Monotonicity (M): For all pairs $(R_1, \cdots, R_L)$ and $(R'_1, \cdots, R'_L)$ of L-tuples of individual orderings in the domain of a SDR f, which maps them respectively into R and R', monotonicity holds iff $\forall\, x,y \in S$:

$$[\forall i : (xP_iy \longrightarrow xP'_iy) \text{ and } (xI_iy \longrightarrow xR'_iy)]$$
$$\longrightarrow [(xPy \longrightarrow xP'y) \text{ and } (xIy \longrightarrow xR'y)].$$

<u>Definition 5</u>: Weak Monotonicity (WM): For all pairs $(R_1,\cdots,R_L)$ and $(R'_1,\cdots,R'_L)$ of L-tuples of individual orderings in the domain of a SDR f, which maps them respectively into R and R', WM holds iff $\forall$ x,y $\in$ S:

$$\Big[\forall k \in N : (\forall i \neq k: (xR_i y \longleftrightarrow xR'_i y) \text{ and } (yR_i x \longleftrightarrow yR'_i x))$$

and $$((xP_k y \longrightarrow xP'_k y) \text{ and } (xI_k y \longrightarrow xR'_k y))\Big] \longrightarrow$$

$$\Big[(xPy \longrightarrow xP'y) \text{ and } (xIy \longrightarrow xR'y)\Big].$$

<u>Remark 1</u>: It can be easily seen that in the presence of conditions U and I, M and WM are equivalent. See Pattanaik [3].

<u>Definition 6</u>: R is a strong ordering iff R is an ordering and $\forall$ x,y $\in$ S : xIy $\longleftrightarrow$ x=y.

<u>Definition 7</u>: R is a null ordering iff $\forall$ x,y $\in$ S : xIy.

The set of all logically possible orderings of the alternatives in the set S will be denoted by π. The set of all logically possible strong orderings of the alternatives in the set S will be denoted by π'. Similarly, π'' will denote the set of all logically possible strong orderings and the null ordering of the alternatives in the set S.

In this terminology unrestricted domain (U) means that every element (situation) belonging to $\pi \times \cdots \times \pi$ (L times) is in the domain of f.

<u>Definition 8</u>: An element x in S is a best element of S with respect to a binary relation R iff

$\forall y : (y \in S \longrightarrow xRy)$.

The set of best elements in S is called its choice set and is denoted by $C(S,R)$.

The society is assumed to adopt the following choice procedure. For every configuration of individual orderings over S social decision rule f assigns a unique reflexive and connected R over S. For every reflexive and connected R over S function C assigns a unique subset of S. C is the function which selects the best elements of S according to the binary preference relation R over S. If the choice set contains exactly one element then that alternative becomes the final outcome. When the choice set contains more than one element we would assume that a random mechanism is employed to select one element from the choice set, such that the probability of any particular element of the choice set being selected is 1/m where m is the number of elements in the choice set. If the choice set is empty we assume that a distinguished alternative $x_o \notin S$ is selected.

The lottery corresponding to choice set C will be denoted by C*. The set of all possible outcomes will be denoted by S*. We will assume that every individual i has an ordering R_i^* defined over S*. Throughout this work we assume that the domain of f is such that for every individual i any logically possible ordering of S is admissible. Now we state the corresponding assumption with respect to set S*. We will assume that for every individual i every logically possible ordering of S* (R_i^*) is admissible which satisfies the following two conditions.

(1) The restriction of R_i^* over S must agree with R_i.

(2) The R_i^* must be consistent with the expected utility maximization principle.

Once R_i is specified, it induces a quasi-ordering (reflexive and transitive binary preference relation) over S*. The precise manner in which it is done is explained in what follows. Let $C_1^*, C_2^* \in S^* - \{x_o\}$ be two lotteries corresponding respectively to choice sets C_1 and C_2. Let the number of indifference classes (according to individual i's preference ordering) in which elements of $C_1 \cup C_2$ can be divided t. Select one element from each indifference class. Arrange these t elements in the decreasing order of preference, $x_1, \cdots, x_t$. Now, for the lottery C_1^* compute the following cumulative probabilities.

$$p(x \succsim_i x_1) = p_1$$
$$p(x \succsim_i x_2) = p_2$$
$$\vdots$$
$$p(x \succsim_i x_{t-1}) = p_{t-1}$$
$$p(x \succsim_i x_t) = p_t$$

Similarly compute these cumulative probabilities for the lottery C_2^* to be designated by q_j's.

Given individual i's preference ordering over S (R_i) and the expected utility maximization principle, and only these, we can assert $C_1^* R_i^* C_2^*$ ($C_1^*, C_2^* \in S^* - \{x_o\}$) iff

$$(p_1 \geq q_1) \text{ and } (p_2 \geq q_2) \text{ and } \cdots \text{ and } (p_{t-1} \geq q_{t-1}) \text{ and } (p_t \geq q_t)$$

An example may help illustrate the point. Let

$$S = \{x, y, z\}$$
$$S^* - \{x_o\} = \{x, y, z, (x,y)^*, (x,z)^*, (y,z)^*, (x,y,z)^*\}$$
$$R_i = xP_i yP_i z.$$

From R_i and the expected utility maximization principal we can conclude the following,

$xP_i^* y$	$y\ P_i^*\ z$	$(x,y,z)^* P_i^*\ z$
$xP_i^* z$	$(x,y)^*\ P_i^*\ y$	$(x,y)^*\ P_i^*(x,z)^*$
$xP_i^*(x,y)^*$	$y\ P_i^*(y,z)^*$	$(x,y)^*\ P_i^*(y,z)^*$
$xP_i^*(x,z)^*$	$(x,y)^*\ P_i^*\ z$	$(x,y)^*\ P_i^*(x,y,z)^*$
$xP_i^*(y,z)^*$	$(x,z)^*\ P_i^*\ z$	$(x,z)^*\ P_i^*(y,z)^*$
$xP_i^*(x,y,z)^*$	$(y,z)^*\ P_i^*\ z$	$(x,y,z)^*\ P_i^*(y,z)^*$

Out of the 21 pairwise preferences over the set $S^* - \{x_o\}$ 18 are determined. The following three are undetermined,

$$\left[y,\ (x,z)^*\right],\ \left[y,\ (x,y,z)^*\right],\ \left[(x,z)^*,\ (x,y,z)^*\right]$$

If one of these is specified the remaining two will be determined in accordance with the expected utility maximization principle. Suppose $yP_i^*(x,z)^*$. Then it must be the case that $yP_i^*(x,y,z)^*$ and $(x,y,z)^*\ P_i^*\ (x,z)^*$. Similarly,

$$(x,z)^*\ P_i^* y \longrightarrow (x,y,z)^* P_i^* y \text{ and } (x,z)^*\ P_i^*\ (x,y,z)^*$$

$$(x,z)^*\ I_i^* y \longrightarrow y\ I_i^*(x,y,z)^* \text{ and } (x,z)^*\ I_i^*(x,y,z)^*.$$

So, given that R_i is $xP_i yP_i z$, there are precisely three orderings of $S^* - \{x_o\}$ which are compatible with both R_i and the expected utility maximization principle. It can be easily checked that there are exactly 41 orderings of set S^* (R_i^*) which are consistent with both R_i and the expected utility maximization principle.

Let R_i be an ordering of set S. Let $\pi^*_{R_i}$ be the set of all R_i^* consistent with R_i and the expected utility maximization principle. As has already been noted it will be assumed that the domain of f is unrestricted, that is to say, every element of $\pi \times \cdots \times \pi$ (L times) is in the domain of f. The corresponding assumption with respect to set S^* is that every element of $(\bigcup_{R_i \in \pi} \pi^*_{R_i}) \times \cdots \times (\bigcup_{R_i \in \pi} \pi^*_{R_i})$ (L times) will be admissible.

Throughout this work we shall denote individual i's true preference ordering over the set S by $\bar{R}_i$ and over the set S* by $\bar{R}_i^*$. Now we introduce the notion of stability of social decision rules.

A SDR is defined to be stable iff its structure is such that no individual ever has any incentive to misrepresent his preferences. It can be easily seen that this is equivalent to requiring that every $\langle \bar{R}_i \rangle$ ($\langle \bar{R}_i \rangle$ is an abbreviation for $(\bar{R}_1, \cdots, \bar{R}_L)$) situation be a Nash equilibrium.

Definition 9: A SDR is stable iff every $\langle \bar{R}_i \rangle$ situation is a Nash equilibrium.

Necessity and Sufficiency of Transitivity for Stability

Definition 10: Weak Pareto-criterion (P): $\forall\, x,y \in S$:
$[\forall i : xP_iy] \longrightarrow xPy$.

Definition 11: Pareto-criterion ($\bar{P}$) : $\forall\, x,y \in S$:
$[\forall i : xR_iy$ and $\exists\, i : xP_iy] \longrightarrow xPy$ and
$[\forall i : xI_iy] \longrightarrow xIy$.

Lemma 1: If f satisfies U, $\bar{P}$ and I, then a necessary condition for the stability of f is that it satisfies condition M.

Proof: Let $S = \{x, y, z_1, \cdots, z_{n-2}\}$. Suppose f violates M then in view of remark 1 there exists a pair of alternatives, say, x,y, and two situations $\langle R_i \rangle$ and $\langle R_i' \rangle$ such that for some k,

$$\forall i \neq k : [(xR_iy \longleftrightarrow xR_i'y) \text{ and } (yR_ix \longleftrightarrow yR_i'x)]$$

$$k : [(yP_kx \text{ and } xI_k'y) V (yP_kx \text{ and } xP_k'y) V (xI_ky \text{ and } xP_k'y)] ,$$

however we have

$$(xPy \text{ and } yP'x) \; V \; (xPy \text{ and } xI'y) \; V \; (xIy \text{ and } yP'x) .$$

Suppose (xPy and $yP'x$) and (yP_kx and xI'_ky). Let the restriction of $\langle R_i \rangle$ over $\{x,y\}$ be characterized as follows:
$\forall i \in N_1 : xP_iy$; $\forall i \in N_2 : xI_iy$; $\forall i \in N_3 : yP_ix$, where $\bigcup_{t=1}^{3} N_t = N$ and $k \in N_3$.

Now, let $\langle \bar{R}_i \rangle$ be as follows:

$$\forall i \in N_1 : x\bar{P}_iy\bar{P}_iz_1\bar{P}_i \cdots \bar{P}_i z_{n-2}$$

$$\forall i \in N_2 : x\bar{I}_iy\bar{P}_iz_1\bar{P}_i \cdots \bar{P}_i z_{n-2}$$

$$\forall i \in N_3 : y\bar{P}_ix\bar{P}_iz_1\bar{P}_i \cdots \bar{P}_i z_{n-2}.$$

By condition I and the fact that $\langle \bar{R}_i \rangle$ is identical with $\langle R_i \rangle$ over $\{x,y\}$, we conclude that $\bar{R}$ and R are identical over $\{x,y\}$. This together with condition $\bar{P}$ completely determines $\bar{R}$.

$$\bar{R} = x\bar{P}y\bar{P}z_1\bar{P} \cdots \bar{P} z_{n-2}.$$

Now construct $\langle R^o \rangle$ as follows:

$$\forall i \in N_1 : xP_i^oyP_i^oz_1P_i^o \cdots P_i^o z_{n-2}$$

$$\forall i \in N_2 \cup \{k\} : xI_i^oyP_i^oz_1P_i^o \cdots P_i^o z_{n-2}$$

$$\forall i \in N_3 - \{k\} : yP_i^oxP_i^oz_1P_i^o \cdots P_i^o z_{n-2}.$$

$\langle R_i^o \rangle$ and $\langle R_i' \rangle$ are identical over $\{x,y\}$. So, R^o and R' must be identical over $\{x,y\}$ by condition I. This in conjunction with condition $\bar{P}$ determined R^o completely.

$$R^o = yP^oxP^oz_1P^o \cdots P^o z_{n-2}.$$

If every individual reveals his true preferences we obtain the situation $\langle \bar{R}_i \rangle$ which yields the outcome x. Now, given that every $i \neq k$ is going to reveal his true preferences if individual k's revealed preferences are $R_k^o \neq \bar{R}_k$ then the situation $\langle R_i^o \rangle$ is obtained which yields the outcome y. As individual k prefers y to x, it follows that the situation $\langle \bar{R}_i \rangle$ is not a Nash equilibrium.

For the case (xPy and yP'x) and (yP_kx and xI'_ky), we have shown that there exists an $\langle \bar{R}_i \rangle$ situation which is not a Nash equilibrium. Similarly, the existence of an $\langle \bar{R}_i \rangle$ situation which is not a Nash equilibrium can be demonstrated in each of the remaining 8 cases. This completes the proof.

Now we introduce the following definitions.

Definition 12: A set of individuals V is (N-A)-almost decisive for x against y iff

$$(\forall i \in A : xI_iy \text{ and } \forall i \in V: xP_iy \text{ and } \forall i \in N-A-V: yP_ix) \longrightarrow xPy,$$

where $A \subsetneq N$ and $A \cap V = \varphi$.

Definition 13: A set of individuals V is (N-A)-decisive for x against y iff

$$(\forall i \in A: xI_iy \text{ and } \forall i \in V: xP_iy) \longrightarrow xPy,$$

where $A \subsetneq N$ and $A \cap V = \varphi$.

Definition 14: A set of individual V is minimally (N-A)-decisive for x against y iff it is (N-A)-decisive for x against y and no proper subset of it is (N-A)-decisive for x against y.

Definition 15: A set of individuals V is (N-A)-decisive if it is (N-A)-decisive for every ordered pair of alternatives.

Definition 16: A set of individuals V is minimally (N-A)-decisive iff it is (N-A)-decisive and no proper subset of it is (N-A)-decisive.

Definition 17: R is acyclical over S iff the following holds:

$$\forall x_1, \cdots, x_j \in S: (x_1Px_2 \text{ and } x_2Px_3 \text{ and } \cdots \text{ and } x_{j-2}Px_{j-1} \text{ and } x_{j-1} \text{ and } x_j) \longrightarrow x_1Rx_j.$$

Definition 18: R is quasi-transitive iff $\forall x, y, z \in S$:

$$(xPy \text{ and } yPz) \longrightarrow xPz.$$

Lemma 2: If f satisfies U, I, $\bar{P}$ and M, then a necessary condition for the stability of f is that it always yields acyclic $\bar{R}$.

Proof: Suppose f does not always yields acyclic R, then there exists a situation $\langle R_i \rangle$ such that the choice set C(A,R) is empty for some $A \subset S$ where $A \neq \varphi$. Let A^o be the smallest nonempty subset (or one of the smallest nonempty subsets) of S for which choice set C(A,R) is empty. Let A^o contain m elements. Without any loss of generality assume $A^o = \{x_1, \cdots, x_m\}$. As A^o is the smallest nonempty subset of S for which choice set is empty we must have a P-cycle of mth order. Without any loss of generality assume $x_1 P x_2 P \cdots P\, x_{m-1} P x_m P x_1$.

Let the restriction of $\langle R_i \rangle$ over $\{x_m, x_1\}$ be characterized as follows : $\forall i \in N_1 : x_m P_i x_1$; $\forall i \in N_2 : x_1 P_i x_m$; $\forall i \in N_3 : x_1 I_i x_m$, where $\overset{3}{\underset{t=1}{U}} N_t = N$. As $x_m P x_1$ by condition $\bar{P}$, N_1 must be nonempty. In view of condition M, N_1 is a $(N-N_3)$-decisive set for x_m against x_1. Hence as a consequence of condition $\bar{P}$ there exists a nonempty set $N_1' \subset N_1$ which is minimally $(N-N_3)$-decisive for x_m against x_1.

Now, let $\langle \bar{R}_i \rangle$ be as follows:

(a) $\forall i \in N : x_1, \cdots, x_m\ \bar{P}_i\ x_{m+1}\ \bar{P}_i \cdots \bar{P}_i\ x_n$

(b) $\forall i \in N_1' \cup N_2 \cup N_3 : (x_k \bar{R}_i x_\ell) \longleftrightarrow (x_k R_i x_\ell)$, for all $x_k, x_\ell \in A^o$.

(c) $\forall i \in N_1 - N_1' : (x_k \bar{R}_i x_\ell) \longleftrightarrow (x_k R_i x_\ell)$, for all $x_k, x_\ell \in A^o - \{x_1\}$.

(d) $\forall i \in N_1 - N_1' : x_1\ \bar{P}_i\ x_2, \cdots, x_m$.

By condition $\bar{P}$, $x_1, \cdots, x_m\ \bar{P}\ x_{m+1}\ \bar{P} \cdots \bar{P}\ x_n$. So the choice set $C(S,\bar{R})$ does not contain any x_k, $m+1 \leq k \leq n$. As $\langle R_i \rangle$ and $\langle \bar{R}_i \rangle$ are identical over $A^o - \{x_1\}$ we must have as a consequence of condition I, $x_2 \bar{P} x_3 \bar{P} \cdots \bar{P}\ x_{m-1}\ \bar{P}\ x_m$. Therefore no x_k, $3 \leq k \leq m$, belongs to $C(S,\bar{R})$.

Now, ($\forall i : x_1 P_i x_2 \longrightarrow x_1 \bar{P}_i x_2$) and ($\forall i: x_1 I_i x_2 \longrightarrow x_1 \bar{R}_i x_2$). As $x_1 P x_2$, $x_1 \bar{P} x_2$ must obtain by condition M. So x_2 does not belong to $C(S,\bar{R})$. We have ($\forall i \in N_1' : x_m \bar{P}_i x_1$) and ($\forall i \in N_3: x_1 \bar{I}_i x_m$), so by the $(N-N_3)$ decisiveness of N_1' for x_m against x_1 we obtain $x_m \bar{P} x_1$. Thus x_1 does not belong to $C(S,\bar{R})$. Hence $C(S,\bar{R})$ is empty and therefore the outcome for the situation $\langle \bar{R}_i \rangle$ is x_o.

Now assume that individual $j \in N_1'$ prefers x_1 to x_o. Construct $\langle R_i' \rangle$ as follows:

(a) $\forall i \in N - \{j\} : R_i' = \bar{R}_i$

(b) $(x_k R_j' x_\ell) \longleftrightarrow (x_k \bar{R}_j x_\ell)$, for all $x_k, x_\ell \in S - \{x_1\}$.

(c) $x_1 P_j' x_k$, $k = 2, \cdots n$.

$\langle R_i' \rangle$ and $\langle \bar{R}_i \rangle$ are identical over $S - \{x_1\}$. Hence we must have $x_2 P' \cdots P' x_m P' x_{m+1} P' \cdots P' x_n$ as a consequence of condition I. Therefore

no x_k, $3 \leq k \leq n$, belongs to $C(S,R')$. (A)

Now ($\forall i: x_1 \bar{P}_i x_2 \longrightarrow x_1 P_i' x_2$) and ($\forall i: x_1 \bar{I}_i x_2 \longrightarrow x_1 R_i' x_2$). As $x_1 \bar{P} x_2$ we must have $x_1 P' x_2$ by condition M. So, x_2 does not belong to $C(S,R')$. (B)

Now for all x_k, $3 \leq k \leq m-1$, we must have $x_1 \bar{R} x_k$. Suppose not, then for some k we would have $x_k P x_1$ which gives a kth order $(k < m)$ P-cycle, $x_1 P \cdots P x_k P x_1$. However this contradicts the assumption that A^o is the smallest nonempty subset of S for which $C(A,R)$ is empty. This contradiction establishes the assertion made above. It can be checked that we have for all x_k, $3 \leq k \leq m-1$,

($\forall i: x_1 P_i x_k \longrightarrow x_1 P_i' x_k$) and ($\forall i : x_1 I_i x_k \longrightarrow x_1 R_i' x_k$).

So by condition M we must have

$x_1 R' x_k$, $3 \leq k \leq m-1$. (C)

We have $\forall i : x_1 P_i' x_k$, $m+1 \leq k \leq n$. So by condition $\bar{P}$ we obtain

$$x_1 P' x_k \ , \ m+1 \leq k \leq n. \qquad (D)$$

The restriction of $\langle R_i' \rangle$ over $\{ x_m, x_1 \}$ is as follows:

$\forall i \in N_1' - \{ j \} : x_m P_i' x_1$; $\forall i \in N_3 : x_m I_i' x_1$;

$\forall i \in N - (N_1' - \{ j \}) - N_3 : x_1 P_i' x_m$. Suppose $x_m P' x_1$.

Then $N_1' - \{ j \}$ is a $(N-N_3)$-decisive set for x_m against x_1 in view of condition M. However this contradicts the fact that N_1' is a minimal $(N-N_3)$- decisive set for x_m against x_1. Therefore we must have $x_1 R' x_m$. This together with (B), (C), and (D), establishes that x_1 belongs to $C(S,R')$. From (A) and (B) no x_k, $2 \leq k \leq n$, belongs to $C(S,R')$. Thus $C(S,R') = \{ x_1 \}$. Therefore situation $\langle R_i' \rangle$ yields the outcome x_1.

As individual j prefers x_1 to x_o, situation $\langle \bar{R}_i \rangle$ is vulnerable to the situation $\langle R_i' \rangle$, and therefore $\langle \bar{R}_i \rangle$ is not a Nash equilibrium. This proves the necessity of acyclicity for stability.

<u>Lemma 3</u>: Let S be a 3-element set of alternatives and let f satisfy conditions U,I,M, and always yields acyclic R. Then, if there exists a a situation which violates quasi-transitivity then there exists a situation belonging to $(\pi'')^L$ which violates quasi-transitivity [3/].

<u>Proof</u>: Let $S = \{ x,y,z \}$. Let $\langle R_i \rangle$ violates quasi-transitivity. Without any loss of generality assume xPy and yPz and $\sim$(xPz). $\sim$(xPz) is equivalent to (xIz v zPx). However, zPx is impossible otherwise acyclicity would be violated. So we must have xIz. Let $\langle R_i \rangle$ be characterized as follows:

$\forall i \in N_1 : xP_i yP_i z$
$\forall i \in N_2 : xP_i zP_i y$
$\forall i \in N_3 : yP_i xP_i z$
$\forall i \in N_4 : yP_i zP_i x$
$\forall i \in N_5 : zP_i xP_i y$
$\forall i \in N_6 : zP_i yP_i x$
$\forall i \in N_7 : xI_i yI_i z$
$\forall i \in N_8 : xP_i yI_i z$
$\forall i \in N_9 : yI_i zP_i x$
$\forall i \in N_{10} : yP_i xI_i z$
$\forall i \in N_{11} : xI_i zP_i y$
$\forall i \in N_{12} : zP_i xI_i y$
$\forall i \in N_{13} : xI_i yP_i z$,

where $\bigcup_{t=1}^{13} N_t = N$.

Construct $\langle R'_i \rangle$ as follows:

(a) $\forall i \in \bigcup_{t=1}^{7} N_t : R'_i = R_i$

(b) $\forall i \in N_8 \cup N_{13} : xP'_i yP'_i z$

(c) $\forall i \in N_9 \cup N_{10} : yP'_i zP'_i x$

(d) $\forall i \in N_{11} \cup N_{12} : zP'_i xP'_i y$.

That is to say, $\langle R'_i \rangle$ is characterized as follows:

$\forall i \in N_1 \cup N_8 \cup N_{13} : xP'_i yP'_i z$
$\forall i \in N_2 : xP'_i zP'_i y$
$\forall i \in N_3 : yP'_i xP'_i z$
$\forall i \in N_4 \cup N_9 \cup N_{10} : yP'_i zP'_i x$
$\forall i \in N_5 \cup N_{11} \cup N_{12} : zP'_i xP'_i y$
$\forall i \in N_6 : zP'_i yP'_i x$
$\forall i \in N_7 : xI'_i yI'_i z$.

Every R'_i is either a strong ordering or null ordering of S. So $\langle R'_i \rangle$ belongs to $(\pi'')^L$. Now we have
$(\forall i: xP_i y \longrightarrow xP'_i y)$ and $(\forall i : xI_i y \longrightarrow xR'_i y)$ and
$(\forall i: yP_i z \longrightarrow yP'_i z)$ and $(\forall i : yI_i z \longrightarrow yR'_i z)$ and
$(\forall i : zP_i x \longrightarrow zP'_i x)$ and $(\forall i : xI_i z \longrightarrow zR'_i x)$.
Given condition M, this in view of the fact that we have xPy and yPz and xIz implies that we must have $xP'y$ and $yP'z$ and $(xI'z \vee zP'x)$.

However, $zP'x$ is impossible otherwise the condition that f always yields acyclic R would be violated. Therefore $xI'z$ must hold. Thus R' is identical to R and violates quasi-transitivity. So we have shown that under the conditions of the lemma, the existence of a situation which violates quasi-transitivity implies the existence of a situation belonging to $(\pi'')^L$ which violates quasi-transitivity.

Lemma 4: If f satisfies $U, I, \bar{P}, M$, and always yields acyclic R then a necessary condition for the stability of f is that it always yields quasi-transitive R.

Proof Let $S = \{x, y, z, w_1, \cdots, w_{n-3}\}$. Suppose f violates quasi-transitivity. Then by lemma 3 there exists a situation $\langle R_i' \rangle$ such that it violates quasi-transitivity over some triple, say, $\{x,y,z\}$ and every individual has either a strong ordering or null ordering over $\{x,y,z\}$. Without any loss of generality assume $xP'y$ and $yP'z$ and $xI'z$. The restriction of $\langle R_i' \rangle$ over $\{x,y,z\}$ and can be characterized as follows:

$\forall i \in N_1 : xP_i'yP_i'z$ $\qquad$ $\forall i \in N_2 : xP_i'zP_i'y$

$\forall i \in N_3 : yP_i'xP_i'z$ $\qquad$ $\forall i \in N_4 : yP_i'zP_i'x$

$\forall i \in N_5 : zP_i'xP_i'y$ $\qquad$ $\forall i \in N_6 : zP_i'yP_i'x$

$\forall i \in N_0 : xI_i'yI_i'z$,

where $\bigcup_{t=0}^{6} N_t = N$.

As $xP'y$, by condition $\bar{P}$, $N_1 \cup N_2 \cup N_5$ is nonempty. In view of condition M, $N_1 \cup N_2 \cup N_5$ is a $(N-N_0)$-decisive set for x against y. Hence as a consequence of condition $\bar{P}$ there exists a nonempty set $Vxy \subset N_1 \cup N_2 \cup N_5$ which is minimally $(N-N_0)$-decisive for x against y. By an analogous argument there exists a nonempty set $Vyz \subset N_1 \cup N_3 \cup N_4$ which is minimally $(N-N_0)$-decisive for y against z.

Now, construct $\langle R_i \rangle$ as follows:

$\forall i \in Vxy \cap Vyz$	: $xP_i yP_i zP_i w_1 P_i \cdots P_i w_{n-3}$
$\forall i \in Vxy - Vyz$	: $zP_i xP_i yP_i w_1 P_i \cdots P_i w_{n-3}$
$\forall i \in Vyz - Vxy$	: $yP_i zP_i xP_i w_1 P_i \cdots P_i w_{n-3}$
$\forall i \in N - Vxy - Vyz - N_o$	: $zP_i yP_i xP_i w_1 P_i \cdots P_i w_{n-3}$
$\forall i \in N_o$	: $xI_i yI_i zP_i w_1 P_i \cdots P_i w_{n-3}$

By condition $\bar{P}$ we have $x,y,z\ Pw_1 P \cdots Pw_{n-3}$. We obtain xPy and yPz as ($\forall i \in Vxy$: $xP_i y$ and $Vi \in N_o$: $xI_i y$) and ($\forall i \in Vyz$: $yP_i z$ and $\forall i \in N_o$: $yI_i z$. Now, ($\forall i$: $zP_i' x \longrightarrow zP_i x$) and ($\forall i$:$zI_i' x \longrightarrow zR_i x$). Therefore in view of condition M (zPx or xIz) must hold as we have $xI'z$. However zPx is impossible because f always yields acyclic R. Therefore xIz holds. Thus R is as follows:

$$x,y,z\ P\ w_1\ P \cdots P\ w_{n-3} : xPy,\ yPz,\ xIz.$$

As has already been argued both Vxy and Vyz are nonempty thanks to condition $\bar{P}$. Furthermore $Vxy \cap Vyz$ must be nonempty otherwise we will get a contradiction as follows. Assume $Vxy \cap V_{yz} = \phi$. Then the restriction of $\langle R_i \rangle$ over $\{x,y,z\}$ becomes as follows $\forall i \in Vxy$: $zP_i xP_i y$, $\forall i \in Vyz$: $yP_i zP_i x$, $\forall i \in N\text{-}Vxy\text{-}Vyz\text{-}N_o$: $zP_i yP_i x$, $\forall i \in N_o$: $xI_i yI_i z$. But then we must have zPx in view of condition $\bar{P}$. However, this contradicts xIz as established above. Hence $Vxy \cap Vyz$ must be nonempty. Let individual $j \in Vxy \cap Vyz$.

Let $\langle \bar{R}_i \rangle$ be as follows:

(a) $\forall i \neq j$: $\bar{R}_i = R_i$

(b) $x\bar{P}_i z\bar{P}_i y\bar{P}_i w_1 \bar{P}_i \cdots \bar{P}_i w_{n-3}$.

$\langle \bar{R}_i \rangle$ and $\langle R_i \rangle$ are identical for all pairs of alternatives $\{s,t\} \neq \{y,z\}$. So by condition I, $\bar{R}$ and R must be identical for all pairs

$\{s,t\} \neq \{y,z\}$. Suppose $y\bar{P}z$. Then $V_{yz} - \{j\}$ is a $(N-N_o)$-decisive set for y against z in view of condition M. However this contradicts the fact that V_{yz} is a minimal $(N-N_o)$-decisive set for y against z. Therefore $y\bar{P}z$ is false, that is to say, $z\bar{R}y$ holds. Thus $\bar{R}$ is as follows:

$$x,y,z\ \bar{P}w_1\bar{P} \cdots \bar{P}w_{n-3};\quad x\bar{P}y,\ z\bar{R}y,\ x\bar{I}z.$$

If every individual i employs the strategy $\bar{R}_i$, the situation $\langle\bar{R}_i\rangle$ results which yields the outcome $(x,z)^*$. Now given that every individual $i \neq j$ is going to use strategy $\bar{R}_i$, if individual j uses the strategy $R_j \neq \bar{R}_j$ the situation $\langle R_i\rangle$ obtains with the outcome x. Individual j prefers the outcome x to $(x,z)^*$. Therefore $\langle\bar{R}_i\rangle$ is not a Nash equilibrium and hence f is unstable. This completes the proof.

Lemma 5: Let f satisfy U, $\bar{P}$, and I. If f always yields quasi-transitive R, then, if a set of individuals V is (N-A)-almost decisive for some ordered pair of alternatives it is (N-A)-decisive for all ordered pairs of alternatives.

Proof: This is a direct generalization of the lemma that Arrow uses in the proof of the General Possibility Theorem. See Arrow [1] and Sen [4].

Lemma 6: Let f satisfy U, $\bar{P}$ and I. If f always yields quasi-transitive R then it implies that for every $A \subsetneq N$, there is a unique minimal (N-A) decisive set.

Proof: This lemma is a straightforward generalization of Gibbard's theorem: For a proof see Guha [2].

Lemma 7: Let S be a 3-element set of alternatives and let f satisfy conditions U, I, M; and always yield quasi-transitive R. Then, if there exists a situation which violates transitivity then there exists a situation belonging to $(\pi'')^L$ which violates transitivity.

Proof: Let $S = \{x,y,z\}$. Let $\langle R_i \rangle$ violate transitivity. Without any loss of generality assume yRz and zRx and $\sim$(yRx). $\sim$(yRx) is equivalent to xPy. Suppose yPz, then we obtain xPz by quasi-transitivity (xPy and yPz $\longrightarrow$ xPz). However, xPz is false, so yPz must be false and therefore yIz is true. By an analogous argument it can be seen that xIz holds. Thus R is as follows: xPy, yIz, xIz.

Let $\langle R_i \rangle$ be characterized as follows:

$\forall i \in N_1$: xP_iyP_iz

$\forall i \in N_2$: xP_izP_iy

$\forall i \in N_3$: yP_ixP_iz

$\forall i \in N_4$: yP_izP_ix

$\forall i \in N_5$: zP_ixP_iy

$\forall i \in N_6$: zP_iyP_ix

$\forall i \in N_7$: xI_iyI_iz

$\forall i \in N_8$: xP_iyI_iz

$\forall i \in N_9$: yI_izP_ix

$\forall i \in N_{10}$: yP_ixI_iz

$\forall i \in N_{11}$: xI_izP_iy

$\forall i \in N_{12}$: zP_ixI_iy

$\forall i \in N_{13}$: xI_iyP_iz

where $\bigcup_{t=1}^{13} N_t = N$.

Now, construct the situation $\langle R_i' \rangle$ as follows:

(a) $\forall i \in \bigcup_{t=1}^{7} N_t$: $R_i' = R_i$

(b) $\forall i \in N_8 \cup N_{13}$: $xP_i'yP_i'z$

(c) $\forall i \in N_9 \cup N_{10}$: $yP_i'zP_i'x$

(d) $\forall i \in N_{11} \cup N_{12}$: $zP_i'xP_i'y$.

So $\langle R_i' \rangle$ is characterized as follows:

$\forall i \in N_1 \cup N_8 \cup N_{13}$: $xP_i'yP_i'z$ $\quad$ $\forall i \in N_5 \cup N_{11} \cup N_{12}$: $zP_i'xP_i'y$

$\forall i \in N_2$: $xP_i'zP_i'y$ $\quad$ $\forall i \in N_6$: $zP_i'yP_i'x$

$\forall i \in N_3$: $yP_i'xP_i'z$ $\quad$ $\forall i \in N_7$: $xI_i'yI_i'z$.

$\forall i \in N_4 \cup N_9 \cup N_{10}$: $yP_i'zP_i'x$

Every R_i' is either a strong ordering or null ordering of S. So $\langle R_i' \rangle$ belongs to $(\pi'')^L$. Now we have

$(\forall i : xP_iy \longrightarrow xP_i'y)$ and $(\forall i : xI_iy \longrightarrow xR_i'y)$ and

$(\forall i : yP_iz \longrightarrow yP_i'z)$ and $(\forall i : yI_iz \longrightarrow yR_i'z)$ and

$(\forall i : zP_ix \longrightarrow zP_i'x)$ and $(\forall i : zI_ix \longrightarrow zR_i'x)$.

Given condition M, this in view of the fact that we have xPy and yIz and xIz implies that we must have xP'y and (yP'z v yI'z) and (zP'x v xI'z). Suppose yP'z, then we obtain xP'z by quasi-transitivity (xP'y and yP'z $\longrightarrow$ xP'z). As xP'z is false it follows that yP'z must be false and hence yI'z holds. By an analogous argument xI'z holds. Thus R' = (xP'y, yI'z, xI'z), which violates transitivity. This completes the proof of the lemma.

<u>Lemma 8</u>: If f belongs to the class of functions which satisfy U, I, $\bar{P}$, M, and always yield quasi-transitive R then a necessary condition for the stability of f is that it always yields transitive R.

<u>Proof</u>: Let S = $\{x, y, z, w_1, \cdots, w_{n-3}\}$. Suppose f violates transitivity. Then by Lemma 7 there exists a situation $\langle R_i' \rangle$ such that it violates transitivity over some triple, say, $\{x,y,z\}$ and every individual has either a strong ordering or null odering over $\{x,y,z\}$. Without any loss of generality assume xP'y and yI'z and xI'z. The restriction of $\langle R_i' \rangle$ over $\{x,y,z\}$ can be characterized as follows:

$\forall i \in N_1 : xP'_i yP'_i z$

$\forall i \in N_2 : xP'_i zP'_i y$

$\forall i \in N_3 : yP'_i xP'_i z$

$\forall i \in N_4 : yP'_i zP'_i x$

$\forall i \in N_5 : zP'_i xP'_i y$

$\forall i \in N_6 : zP'_i yP'_i x$

$\forall i \in N_o : xI'_i yI'_i z,$

where $\bigcup_{t=o}^{6} N_t = N.$

We have $\forall i \in N_1 \cup N_2 \cup N_5 : xP'_i y$ and $\forall i \in N_3 \cup N_4 \cup N_6 : yP'_i x$ and $\forall i \in N_o : xI'_i y$. This yields $xP'y$. Therefore $N_1 \cup N_2 \cup N_5$ is $(N-N_o)$-almost decisive for (x,y). By lemma 5 this implies that $N_1 \cup N_2 \cup N_5$ is a $(N-N_o)$-decisive set. Now, by lemma 6 there exists a unique minimal $(N-N_o)$-decisive set V. Therefore $V \subset N_1 \cup N_2 \cup N_5$. Now $N_1 \cap V$ must be nonempty. Suppose not, then $V \subset N_2 \cup N_5$. As $\forall i \in N_2 \cup N_5 : zP'_i y$ and $\forall i \in N_o : zI'_i y$, we must have $zP'y$ by the $(N-N_o)$ decisiveness of V. However $zP'y$ is false. This proves that $N_1 \cap V$ is nonempty. By an analogous argument $N_5 \cap V$ can be shown to be nonempty.

Now, let situation $\langle \bar{R}_i \rangle$ be as follows.

$\forall i \in (N_1 \cap V) \cup (N_2 \cap V) : x\bar{P}_i y\bar{P}_i z\bar{P}_i w_1\bar{P}_i \ldots \bar{P}_i w_{n-3}$

$\forall i \in N_5 \cap V : z\bar{P}_i x\bar{P}_i y\bar{P}_i w_1\bar{P}_i \ldots \bar{P}_i w_{n-3}$

$\forall i \in N-V-N_o : y\bar{P}_i z\bar{P}_i x\bar{P}_i w_1\bar{P}_i \ldots \bar{P}_i w_{n-3}$

$\forall i \in N_o : x\bar{I}_i y\bar{I}_i z\bar{P}_i w_1\bar{P}_i \ldots \bar{P}_i w_{n-3}.$

Let $N'_1 = (N_1 \cap V) \cup (N_2 \cap V)$, $N'_2 = N_5 \cap V$, $N'_3 = N-V-N_o$. As $\forall i \in V: x\bar{P}_i y$ and $\forall i \in N_o : x\bar{I}_i y$, we must have $x\bar{P}y$. Suppose $y\bar{P}z$. This implies that $N'_1 \cup N'_3$ is an $(N-N_o)$-almost decisive set for (y,z). Then by lemma 5, $N'_1 \cup N'_3$ is an $(N-N_o)$-decisive set. Hence there exists a minimal $(N-N_o)$-decisive set $V' \subset N'_1 \cup N'_3$. As $V' \cap N'_2 = \phi$ and $N'_2 \neq \phi$, it follows that $V \neq V'$. However this contradicts the result of lemma 6 that there is a unique minimal $(N-N_o)$-decisive set. Therefore $y\bar{P}z$ is false.

Now suppose $z\bar{P}y$. Then N_2' is an $(N-N_0)$-almost decisive set for (z,y) and hence a $(N-N_0)$-decisive set in view of lemma 5. Therefore there exists a minimal $(N-N_0)$-decisive set $V' \subset N_2'$. As $V' \cap N_1' = \phi$ and $N_1' \neq \phi$, it follows that $V \neq V'$. However it contradicts the result of lemma 6 that there is a unique minimal $(N-N_0)$-decisive set. Therefore $z\bar{P}y$ is false. Hence by the connectedness of $\bar{R}$, $y\bar{I}z$ must hold. By an analogous argument it can be shown that $x\bar{I}z$ must hold. Thus $\bar{R}$ is as follows:

$$x,y,z\ \bar{P}w_1\bar{P} \cdots \bar{P}w_{n-3}\ ;\ x\bar{P}y,\ y\bar{I}z,\ x\bar{I}z.$$

Let $j \in N_1'$ and assume that individual j prefers $(x,y,z)^*$ to $(x,z)^*$. Construct the situation $\langle R_i \rangle$ as follows:

(a) $\forall i \neq j : R_i = \bar{R}_i$

(b) $yP_jxP_jzP_jw_1P_j \cdots P_jw_{n-3}$.

$\langle \bar{R}_i \rangle$ and $\langle R_i \rangle$ are identical for all pairs $\{s,t\} \neq \{x,y\}$. Therefore $\bar{R}$ and R must be identical for all pairs $\{s,t\} \neq \{x,y\}$. Suppose xPy, then $V - \{j\}$ is an $(N-N_0)$-decisive set in view of Lemma 5. However this contradicts the fact that V is a minimal $(N-N_0)$-decisive set. Therefore xPy is false. Next suppose yPx. This means that $N_3' \cup \{j\}$ is an $(N-N_0)$ - decisive set. Therefore there exists a minimal $(N-N_0)$-decisive set $V' \subset N_3' \cup \{j\}$. As $N_3' \cup \{j\} \cap N_2' = \phi$ and $N_2' \neq \phi$, it follows that $V \neq V'$. This contradicts the result of lemma 6 that there is a unique minimal $(N-N_0)$-decisive set. Therefore yPx is false. Hence xIy must hold. Thus R is as follows:

$$x,y,z\ P\ w_1\ P \cdots P\ w_{n-3}\ ;\ xIy,\ yIz,\ xIz.$$

$\langle \bar{R}_i \rangle$ yields the outcome $(x,z)^*$ and $\langle R_i \rangle$ the outcome $(x,y,z)^*$. As individual j prefers $(x,y,z)^*$ to $(x,z)^*$, it follows that $\langle \bar{R}_i \rangle$ is not a Nash equilibrium being vulnerable to $\langle R_i \rangle$. This completes the proof.

Lemma 9: Let f satisfy U, I, and $\bar{P}$. If f always yields transitive R then for every $A \subsetneq N$ there exists a unique minimal (N-A)-decisive set which consists of a single individual.

Proof: By lemma 6 for every $A \subsetneq N$ a unique minimal (N-A)-decisive set is implied. So the only thing that we have to prove is that for every $A \subsetneq N$, the unique minimal (N-A)-decisive set consists of a single individual. Suppose that the lemma is false, then for some A the unique minimal (N-A)-decisive set V must consist of at least two individuals. Partition V into V_1 and V_2 where V_1 consists of a single individual. Now consider the following configuration of preferences.

$\forall i \in V_1$: xP_iyP_iz

$\forall i \in V_2$: zP_ixP_iy

$\forall i \in N-A-V$: zP_iyP_ix

$\forall i \in A$: xI_iyI_iz.

By the (N-A) decisiveness of V and the fact that ($\forall i \in A : xI_iy$) and ($\forall i \in V : xP_iy$) we obtain xPy. Suppose zPy. Then $N-A-V_1$ is (N-A)-almost decisive for (z,y). This of course implies that $N-A-V_1$ is (N-A)-decisive by lemma 5. Hence there exists a minimal (N-A)-decisive set $V' \subset N-A-V_1$. As $V_1 \neq \phi$ and $V' \cap V_1 = \phi$, it follows that V and V' are not identical. However, this contradicts the result of lemma 6 that there must exist a unique minimal (N-A)-deicisive set. So zPy must be false. Hence yRz holds. Now, xPy and yRz $\longrightarrow$ xPz, by transitivity. Thus V_1 is an (N-A)-almost decisive set for (x,z). Therefore, by lemma 5, V_1 is (N-A)decisive. But this contradicts the supposition that V is a minimal (N-A)-decisive set. This contradiction establishes the lemma.

Theorem: If f belongs to the class of functions which satisfy U, I, and $\bar{P}$, then monotonicity and transitivity constitute a set of necessary and sufficient conditions for the stability of f.

Proof: The necessity part follows from lemmas 1,2,4 and 8. So here we just prove the sufficiency.

By lemma 9 there exists a unique minimal $(N-\phi)$-decisive set which consists of a single individual, say, i_1. Similarly there exists a unique minimal $(N- i_1)$-decisive set which consists of a single individual, say, i_2; and so on. Let T be the ordered set of all individuals, $T = (i_1, \cdots, i_r, \cdots, i_L)$, such that i_r is minimally $(N- i_1, \cdots, i_{r-1})$-decisive.

If every individual is indifferent among all the alternatives then by condition $\bar{P}$ all alternatives are socially indifferent. Clearly in this situation no individual has any incentive to misrepresent his preferences. Now assume that there exists at least one individual who is not indifferent among all alternatives. Let i_r be the first individual who is not indifferent among all alternatives. By the $(N- i_1, \cdots, i_{r-1})$-decisiveness of i_r no alternative which is not individual i_r's first preference can belong to the choice set. If $\{x_1, \cdots, x_k\}$ is the set of i_r's first preferences then the choice set must be a nonempty subset of $\{x_1, \cdots, x_k\}$. Therefore individual i_r has no incentive to misrepresent his preferences.

Irrespective of individual i_{r+1}'s preference ordering no alternative which is not a first preference in individual i_r's ordering can belong to the choice set. As individual i_{r+1} is $(N- \{i_1, \cdots, i_r\})$decisive, the choice set must be a subset of i_{r+1}'s first preferences over $\{x_1, \cdots, x_k\}$. Thus individual i_{r+1} has no incentive to misrepresent his preferences. By continuing this way we see that no individual i_s, $r \leq s \leq L$, has any incentive to misrepresent his preferences. Individuals i_1 to i_{r-1} have no incentive to misrepresent their preferences as they are indifferent among all alternatives. Therefore f is stable. This completes the proof of the Theorem.

It can be easily seen that if f satisfies U, $\overline{P}$, I, and always yields transitive R then f must be monotonic. In view of this the above theorem can be restated as follows.

Theorem: If f belongs to the class of functions which satisfy U, I, and $\overline{P}$, then transitivity is a necessary and sufficient condition for the stability of f.

Footnotes

* I am greatly indebted to Professor James Friedman for many helpful comments.

1. A social decision rule is weakly resolute iff $\forall x, y \in S: xIy \longrightarrow \forall i : xI_i y$.

2. The proof presents no difficulties but is rather tedious, so we omit it.

3. $(\pi'')^L$ denotes the Cartesian products $\pi'' X \cdots X \pi''$ (L times).

References

1. Arrow, K.J. : Social Choice and Individual Values, Wiley, New York, 1951, 2nd ed. 1963.

2. Guha, A.S.: Neutrality, Monotonicity, and the Right of Veto, Econometrica, 40, 1972.

3. Pattanaik, P.K. : Strategic Voting Without Collusion Under Binary and Democratic Group Decision Rules, Review of Economic Studies, 42, 1975.

4. Sen, A.K.: Collective Choice and Social Welfare, Holden-Day, San Francisco, 1970.