Maximal Conditions for Transitivity under Neutral and Monotonic Binary Social Decision Rules

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I N D I A

Abstract

It is shown that (i) for the class of binary social decision rules satisfying neutrality and monotonicity, strict placement restriction is maximally sufficient for transitivity and (ii) for the class of binary social decision rules satisfying neutrality, monotonicity and Pareto-criterion, placement restriction is maximally sufficient for transitivity.

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The purpose of this paper is to obtain the maximal sufficient condition for transitivity under neutral and monotonic binary social decision rules. In the context of a class of social decision rules, a condition is maximally sufficient for transitivity if it is sufficient for transitivity and is implied by every condition which ensures transitivity under every social decision rule belonging to the class in question. Thus, for a class of social decision rules, the maximal sufficient condition for transitivity is the weakest condition which ensures transitivity under every social decision rule belonging to the class. We show that a condition called strict placement restriction is maximally sufficient for transitivity under the class of neutral and monotonic binary social decision rules. Strict placement restriction is a condition over triples of alternatives and requires (i) that there be an alternative in the triple which in every concerned individual's ordering is uniquely best, or is uniquely medium, or is uniquely worst or (ii) that there be a pair of distinct alternatives over which every individual is indifferent.

We also consider the important subclass of binary social decision rules which satisfy, in addition to the conditions of neutrality and monotonicity, the Pareto-criterion. We show that for this class of social decision rules placement restriction is maximally sufficient for transitivity. The placement restriction, as the name suggests, is a weaker condition than the strict placement restriction. Placement restriction requires (i) that there be an alternative in the triple such that in every concerned individual's ordering it is best, or uniquely medium, or worst or (ii) that there be a pair of distinct alternatives such that every individual is indifferent over the pair.[1]

1. Restrictions on Preferences:

The set of social alternatives will be denoted by S. We assume $\# S = n > 2$. The set of individuals and the number of individuals are designated by L and N respectively. N is assumed to be finite and greater than 2 .

A binary relation R over a set S is (i) reflexive iff $\forall x \in S : x R x$ (ii) connected iff $\forall x, y \in S : (x \neq y \longrightarrow x R y \vee y R x)$ (iii) transitive iff $\forall x, y, z \in S : (x R y \wedge y R z \longrightarrow x R z)$ (iv) an ordering iff it is reflexive, connected and transitive.

Each individual $i \in L$ is assumed to have an ordering R_i over S. The symmetric and asymmetric parts of R_i are denoted by I_i and P_i respectively. The social weak preference relation is denoted by R and its symmetric and asymmetric components by I and P respectively. $N(\ )$ will stand for the number of individuals holding the preferences specified in the parenthesis, and N_k for the number of individuals holding the k-th preference ordering.

A social decision rule (SDR) is a functional relation which for every N-tuple $(R_1, \ldots . R_N)$, to be written $\langle R_i \rangle$ in abbreviated form, of individual orderings (one ordering for each individual) assigns a unique reflexive and connected social weak preference relation R. An SDR satisfies the condition of independence of irrelevant alternatives (I) iff whenever $\langle R_i \rangle$ and $\langle R_i' \rangle$ are identical over $A \subset S$, R and R' are identical over A, where $R = f\langle R_i \rangle$ and $R' = f\langle R_i' \rangle$. An SDR which satisfies condition I will be called a binary SDR. A binary SDR satisfies (i) neutrality (N) iff $\forall\, x,y,z,w \in S : [[\ \forall i : [(x\ R_i\ y \longleftrightarrow z\ R_i'\ w) \wedge (y\ R_i\ x \longleftrightarrow w\ R_i'\ z)]] \longrightarrow [(x\ R\ y \longleftrightarrow z\ R'\ w) \wedge (y\ R\ x \longleftrightarrow w\ R'\ z)]]$ (ii) monotonicity (M) iff $\forall\, x,y \in S : [[\ \forall i : [(x\ P_i\ y \longrightarrow x\ P_i'\ y) \wedge (x\ I_i\ y \longrightarrow x\ R_i'\ y)]]$

$\longrightarrow [(x\ P\ y \longrightarrow x\ P'\ y) \wedge (x\ I\ y \longrightarrow x\ R'\ y)]]$

(iii) Pareto-criterion ($\overline{P}$) iff $\forall\ x,y \in S : [[\ \forall i : x\ R_i\ y \wedge \exists j : x\ P_j\ y \longrightarrow x\ P\ y\ ] \wedge [\ \forall i : x\ I_i\ y \longrightarrow x\ I\ y\ ]]$.

An individual is defined to be concerned with respect to a triple iff he is not indifferent over every pair of alternatives belonging to the triple; otherwise he is unconcerned. For individual i, in the triple $\{x,y,z\}$, x is best iff $(x\ R_i\ y \wedge x\ R_i\ z)$; medium iff $(y\ R_i\ x\ R_i\ z \vee z\ R_i\ x\ R_i\ y)$; worst iff $(y\ R_i\ x \wedge z\ R_i\ x)$; uniquely best iff $(x\ P_i\ y \wedge x\ P_i\ z)$; uniquely medium iff $(y\ P_i\ x\ P_i\ z \vee z\ P_i\ x\ P_i\ y)$; and uniquely worst iff $(y\ P_i\ x \wedge z\ P_i\ x)$.

Now we define two restrictions which specify the permissible sets of individual orderings. Both the restrictions are defined over triples of alternatives.

Placement Restriction (PR): It is satisfied over a triple iff there exists (i) an alternative such that it is best in every R_i or (ii) an alternative such that it is worst in every R_i

or (iii) an alternative such that it is uniquely medium in every concerned R_i or (iv) a pair of distinct alternatives such that every individual is indifferent between the alternatives of the pair. More formally, PR holds over $\{x, y, z\}$ iff there exist distinct $a, b, c \in \{x, y, z\}$ such that $[\forall i : (a\ R_i\ b \wedge a\ R_i\ c) \vee \forall i : (b\ R_i\ a \wedge c\ R_i\ a) \vee \forall i : (\sim(a\ I_i\ b\ I_i\ c) \longrightarrow (b\ P_i\ a\ P_i\ c \vee c\ P_i\ a\ P_i\ b)) \vee \forall i : a\ I_i\ b\ ]$.

Strict Placement Restriction (SPR): It is satisfied over a triple iff there exists (i) an alternative such that it is uniquely best in every concerned R_i or (ii) an alternative such that it is uniquely worst in every concerned R_i or (iii) an alternative such that it is uniquely medium in every concerned R_i or (iv) a pair of distinct alternatives such that every individual is indifferent between the alternatives of the pair, i.e., SPR holds over $\{x, y, z\}$ iff $\exists$ distinct $a, b, c \in \{x, y, z\}$ such that $[\forall$ concerned $i : (a\ P_i\ b \wedge a\ P_i\ c) \vee \forall$ concerned $i : (b\ P_i\ a \wedge c\ P_i\ a) \vee \forall$ concerned $i : (b\ P_i\ a\ P_i\ c \vee c\ P_i\ a\ P_i\ b) \vee \forall\ i : a\ I_i\ b]$.

2. Conditions for Transitivity:

Theorem 1 : For every neutral and monotonic binary social decision rule, a sufficient condition for transitivity of social weak preference relation is that the strict placement restriction holds over every triple of alternatives.

Proof: Suppose transitivity is violated. Then there are $x, y, z \in S$ such that $x \; R \; y \wedge y \; R \; z \wedge z \; P \; x$.

For any $a, b \in S$ we have, as a consequence of condition N,

$$\forall i : a \; I_i \; b \longrightarrow a \; I \; b \qquad (1)$$

From (1) and condition M we conclude,

$$\forall i : a \; R_i \; b \longrightarrow a \; R \; b \qquad (2)$$

(2) is equivalent to ,

$$\sim a \; R \; b \longrightarrow \sim (\forall i : a \; R_i \; b) \qquad (3)$$

(3) is equivalent to ,

$$b \; P \; a \longrightarrow \exists i : b \; P_i \; a \qquad (4)$$

Therefore we conclude,

$$z \; P \; x \longrightarrow \exists i : z \; P_i \; x \qquad (5)$$

Now, $\forall i : [(z\ P_i\ x \longrightarrow y\ P_i\ x) \wedge (z\ I_i\ x \longrightarrow y\ R_i\ x)]$ must be false, otherwise we would obtain, as a consequence of $z\ P\ x$ and conditions N and M, $y\ P\ x$ which is false. Therefore,

$$\exists i : (z\ P_i\ x\ R_i\ y \vee z\ I_i\ x\ P_i\ y) \qquad (6)$$

Similarly, $\forall i : [(z\ P_i\ x \longrightarrow z\ P_i\ y) \wedge (z\ I_i\ x \longrightarrow z\ R_i\ y)]$ is false. So,

$$\exists i : (y\ R_i\ z\ P_i\ x \vee y\ P_i\ z\ I_i\ x) \qquad (7)$$

(5), (6), and (7) imply that the set of individual orderings over $\{x, y, z\}$ must contain at least one of the following sets of orderings :

(A) $z\ P_i\ x\ R_i\ y$
$y\ R_i\ z\ P_i\ x$

(B) $z\ P_i\ x\ R_i\ y$
$y\ P_i\ z\ I_i\ x$

(C) $z\ I_i\ x\ P_i\ y$
$y\ R_i\ z\ P_i\ x$

(D) $z\ I_i\ x\ P_i\ y$
$y\ P_i\ z\ I_i\ x$
$z\ P_i\ x$

As each of these sets violates SPR, it follows that SPR is a sufficient condition for transitivity.

Lemma 1 : A set of individual orderings over $\{x, y, z\}$ violates strict placement restriction iff it contains one of the following 8 sets of orderings, except for a formal interchange of alternatives :

(i) 1. $x P_i y P_i z$
2. $y P_i z P_i x$

(ii) 1. $x P_i y P_i z$
2. $z I_i x P_i y$

(iii) 1. $x P_i y P_i z$
2. $y P_i z I_i x$

(iv) 1. $x P_i y P_i z$
2. $z P_i x I_i y$

(v) 1. $x P_i y P_i z$
2. $y I_i z P_i x$

(vi) 1. $x P_i y I_i z$
2. $x I_i y P_i z$

(vii) 1. $x P_i y I_i z$
2. $y P_i z I_i x$

(viii) 1. $x I_i y P_i z$
2. $y I_i z P_i x$

<u>Proof</u>: It can be easily checked that there does not exist an alternative such that it is uniquely best in every concerned R_i iff the set of R_i contains one of the following sets, except for a formal interchange of alternatives :

(A) $x\ P_i\ y\ P_i\ z$
$y\ P_i\ z\ P_i\ x$

(B) $x\ P_i\ y\ P_i\ z$
$y\ P_i\ x\ P_i\ z$

(C) $x\ P_i\ y\ P_i\ z$
$z\ P_i\ y\ P_i\ x$

(D) $x\ P_i\ y\ P_i\ z$
$y\ P_i\ z\ I_i\ x$

(E) $x\ P_i\ y\ P_i\ z$
$z\ P_i\ x\ I_i\ y$

(F) $x\ P_i\ y\ I_i\ z$
$y\ P_i\ z\ I_i\ x$

(G) $x\ I_i\ y\ P_i\ z$

A, D, E and F are the same as (i), (iii), (iv) and (vii) respectively. Therefore it suffices to consider only B, C and G.

(B) would violate SPR iff a concerned ordering in which z is not uniquely worst is included. With the inclusion of such an ordering one of the 8 sets is contained in the set of R_i. (C) would violate SPR iff a concerned ordering in which y is not uniquely medium is included. With the inclusion of the required ordering we see that one of the 8 sets is included in the set of R_i. Finally, (G) would violate SPR only if a concerned ordering in which z is not uniquely worst is included. Excepting the case when the ordering included is $z\ P_i\ x\ I_i\ y$, in all other cases SPR is violated and one of the 8 sets is contained in the set of R_i. In the case of inclusion of $z\ P_i\ x\ I_i\ y$, SPR is violated iff an ordering in which $\sim(x\ I_i\ y)$ obtains is included. With the inclusion of the required ordering the set of R_i contains one of the 8 sets. Proof of the lemma is completed by noting that all the 8 sets violate SPR .

Theorem 2: For every restriction C on preferences, if C is a sufficient condition for transitivity of every neutral and monotonic binary social decision rule then $(C \longrightarrow \text{SPR})$.

Proof: For proving the theorem it suffices to show that there exists an SDR satisfying conditions I, M and N for which SPR is necessary for transitivity.

Let $S = \{x, y, z\}$ and $L = \{1, 2, 3, 4\}$. The SDR is characterized as follows :

(1) $\forall\ a, b \in S : a\ P\ b \longleftrightarrow \forall i : a\ R_i\ b \wedge N\ (a\ P_i\ b) \geqslant 3$

(2) $\forall\ a, b \in S : a\ R\ b \longleftrightarrow \sim b\ P\ a$.

As, by lemma 1, SPR is violated iff the set of R_i contains one of the 8 sets mentioned in the statement of the lemma, for demonstrating the necessity of SPR for this SDR it suffices to show that for each of the 8 sets there exists an assignment of individuals which results in intransitive social weak preference relation.

For (i), (ii), (iii) and (vi) take, $N_1 = N_2 = 2$; and for (iv), (v), (vii) and (viii), $N_1 = 3$, $N_2 = 1$. This results for (i), (iii), (v) and (viii) in $x\ I\ y \wedge y\ P\ z \wedge x\ I\ z$, for (ii) and (iv) in $x\ P\ y \wedge y\ I\ z \wedge x\ I\ z$, and for (vi) and (vii) in $x\ I\ y \wedge y\ I\ z \wedge x\ P\ z$.

Corollary 1: For every restriction C on preferences, if C is a sufficient condition for transitivity of every neutral and monotonic binary social decision rule satisfying anonymity then (C $\longrightarrow$ SPR).

Proof: Immediate in view of the fact that the SDR in the proof of theorem 2 satisfies anonymity.

Theorem 3 : For every neutral and monotonic binary social decision rule satisfying the Pareto-criterion, a sufficient condition for transitivity of social weak preference relation is that the placement restriction holds over every triple of alternatives.

Proof: Suppose transitivity is violated. Then there are $x, y, z \in S$ such that $x \, R \, y \wedge y \, R \, z \wedge z \, P \, x$.

As neutrality and monotonicity are satisfied, as in the proof of theorem 1, we conclude that

$$\exists i : \; z \, P_i \, x \tag{1}$$

$$\exists i : [\, z \, P_i \, x \, R_i \, y \;\; \vee \;\; z \, I_i \, x \, P_i \, y \,] \tag{2}$$

$$\exists i : [\, y \, R_i \, z \, P_i \, x \;\; \vee \;\; y \, P_i \, z \, I_i \, x \,] \tag{3}$$

$$(2) \longrightarrow \exists i : z \, P_i \, y \tag{4}$$

As $y \, R \, z$ holds and Pareto-criterion is satisfied, we conclude that

$$(4) \longrightarrow \exists i : \; y \, P_i \, z \tag{5}$$

$$(3) \longrightarrow \exists i : \; y \, P_i \, x \tag{6}$$

As we have $x \, R \, y$ and Pareto-criterion holds, it follows that

$$(6) \longrightarrow \exists i : \; x \, P_i \, y \tag{7}$$

(1) through (7) imply that PR is violated. Thus violation of transitivity implies violation of PR, i.e., PR is a sufficient condition for transitivity.

Lemma 2 : A set of individual orderings over $\{x, y, z\}$ violates PR iff it contains one of the following 10 sets of orderings, except for a formal interchange of alternatives :

(i) 1. $x \, P_i \, y \, P_i \, z$
2. $y \, P_i \, z \, P_i \, x$

(ii) 1. $x \, P_i \, y \, P_i \, z$
2. $z \, P_i \, x \, I_i \, y$

(iii) 1. $x \, P_i \, y \, P_i \, z$
2. $y \, I_i \, z \, P_i \, x$

(iv) 1. $x \, P_i \, y \, P_i \, z$
2. $z \, P_i \, y \, P_i \, x$
3. $y \, P_i \, x \, I_i \, z$

(v) 1. $x \, P_i \, y \, P_i \, z$
2. $z \, P_i \, y \, P_i \, x$
3. $x \, I_i \, z \, P_i \, y$

(vi) 1. $x \, P_i \, y \, I_i \, z$
2. $y \, P_i \, z \, I_i \, x$
3. $z \, P_i \, x \, I_i \, y$

(vii) 1. $x \, I_i \, y \, P_i \, z$
2. $y \, I_i \, z \, P_i \, x$
3. $z \, I_i \, x \, P_i \, y$

(viii) 1. $x \, P_i \, y \, P_i \, z$
2. $y \, P_i \, z \, I_i \, x$
3. $z \, I_i \, x \, P_i \, y$

(ix) 1. $x \, P_i \, y \, I_i \, z$
2. $y \, P_i \, z \, I_i \, x$
3. $z \, I_i \, x \, P_i \, y$

(x) 1. $x \, I_i \, y \, P_i \, z$
2. $y \, P_i \, z \, I_i \, x$
3. $z \, I_i \, x \, P_i \, y$

Proof: It can be easily seen that there do not exist distinct $a, b \in \{x, y, z\}$ such that for $\forall i$: $a \, I_i \, b$ iff the set of R_i

contains one of the following 4 sets, except for a formal interchange of alternatives :

(A) $x P_i y I_i z$
$y P_i z I_i x$

(B) $x I_i y P_i z$
$y I_i z P_i x$

(C) $x P_i y I_i z$
$x I_i y P_i z$

(D) $x P_i y P_i z$

(A) would violate PR iff an ordering in which z is not worst is included. With the inclusion of such an ordering the set of R_i contains one of the 10 sets. (B) would violate PR iff an ordering in which y is not best is included. With the inclusion of required ordering one of the 10 sets is contained in the set of R_i. (C) would violate PR only if an ordering in which z is not worst is included. In all cases, excepting when $x P_i z P_i y$ or $x I_i z P_i y$ is included, PR is violated and one of the 10 sets is contained in the set of R_i. If $x P_i z P_i y$ or $x I_i z P_i y$ is included then PR is violated iff an ordering in which x is not best is included. With the inclusion of required ordering the set of R_i contains one of the 10 sets.

Finally, consider (D). PR would be violated only if an ordering in which x is not best is included. With the inclusion of such an ordering, excepting the cases when $y P_i x P_i z$ or $z P_i y P_i x$ or $y P_i x I_i z$ is included, PR is violated and one of the 10 sets is contained in the set of R_i. If $y P_i x P_i z$ or $y P_i x I_i z$ is included then PR is violated iff an ordering in which z is not worst is included, and in the case of inclusion of $z P_i y P_i x$ PR is violated iff a concerned ordering in which y is not uniquely medium is included. In all cases, with the inclusion of required ordering one of the 10 sets is contained in the set of R_i. This establishes the lemma in view of the fact that all 10 sets violate PR .

Theorem 4 : For every restriction C on preferences, if C is a sufficient condition for transitivity of every Pareto-inclusive neutral and monotonic binary social decision rule then $(C \longrightarrow PR)$.

Proof: For proving the theorem it suffices to show that there exists an SDR satisfying conditions I, M, N and $\overline{P}$ for which PR is necessary for transitivity.

Let $S = \{x, y, z\}$ and $L = \{1, 2, \ldots, N\}$, where N is a multiple of 60. We take SDR to be the two-thirds majority rule which is characterized as follows :

$$\forall a, b \in S : \; a R b \longleftrightarrow N(b P_i a) \leqslant \frac{2}{3} [N(a P_i b) + N(b P_i a)].$$

As, by lemma 2, PR is violated iff the set of R_i contains one of the 10 sets mentioned in the statement of the lemma, for showing the necessity of PR for the above SDR it suffices to show that for each of these sets there exists an assignment of individuals which results in intransitive social weak preference relation.

For (i), (ii) and (iii) take, $N_1 = N_2 = \frac{N}{2}$; for (iv) and (v), $N_1 = \frac{N}{2}$, $N_2 = N_3 = \frac{N}{4}$; for (vi) and (vii), $N_1 = \frac{2}{5} N+1$, $N_2 = \frac{N}{5}$, $N_3 = \frac{2}{5} N-1$; for (viii), $N_1 = N_2 = N_3 = \frac{N}{3}$; for (ix), $N_1 = \frac{N}{2}$, $N_2 = \frac{N}{3}$, $N_3 = \frac{N}{6}$; and for (x), $N_1 = \frac{N}{2}$, $N_2 = \frac{N}{6}$, $N_3 = \frac{N}{3}$. This results in, for (i), (iii) and (iv) in $x I y \wedge y P z \wedge x I z$, for (ii), (v) and (vi) in $x P y \wedge y I z \wedge x I z$, and for (vii) through (x) in $x I y \wedge y I z \wedge x P z$.[2]

Corollary 2 : For every restriction C on preferences, if C is a sufficient condition for transitivity of every Pareto-inclusive neutral and monotonic binary social decision rule satisfying anonymity then $(C \longrightarrow PR)$.

Proof: As the SDR in the proof of theorem 4 satisfies anonymity the result follows immediately.

Footnotes

1. For the classes of social decision rules in question, maximal conditions for quasi-transitivity have been obtained by Sen [5]. Sen has shown that for every monotonic and neutral binary social decision rule value restriction is a sufficient condition for quasi-transitivity, and for every Pareto-inclusive monotonic and neutral binary social decision rule (value restriction or limited agreement) is a sufficient condition for quasi-transitivity. The maximality of these conditions can be easily verified.

2. This demonstrates that placement restriction is necessary and sufficient for transitivity under the two-thirds majority rule. In [4] it is shown that placement restriction is a necessary and sufficient condition for transitivity under every special majority rule, not just under the two-thirds majority rule.

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