Discussion Paper Series

CHARACTERIZATION OF RATIONALITY CONDITIONS IN TERMS OF MINIMAL DECISIVE SETS

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Characterization of Rationality Conditions in Terms of Minimal Decisive Sets

The purpose of this note is to characterize various rationality conditions in terms of minimal decisive sets for the class of social decision rules which satisfy the condition of independence of irrelevant alternatives, strict Pareto-criterion, monotonicity and neutrality.

First, we define the various terms which would be used in the analysis.

Definition 1: A social decision rule (SDR) is a functional relation f such that for any set of N individual orderings $R_1, \cdots, R_N$ (One ordering for each individual), one and only one reflexive and connected social preference relation R is determined,

$$R = f(R_1, \cdots, R_N) \quad .$$

Definition 2: Condition of Independence of Irrelevant Alternatives (I): Let R and R' be the social binary relations determined by f corresponding respectively to two sets of individual orderings, $(R_1, \cdots, R_N)$ and $(R_1', \cdots, R_N')$. If for all pairs of alternatives x,y in a subset A of S, $x R_i y \Longleftrightarrow x R_i' y$ for all i, then $x R y \Longleftrightarrow x R' y$, for all $x,y \in A$.

Definition 3: Monotonicity (M): For all pairs $(R_1, \cdots, R_N)$ and $(R_1', \cdots, R_N')$ of N-tuples of individual orderings in the domain of a SDR f, which maps them respectively into R and R', monotonicity holds iff $\forall x,y \in S$:

$$[\forall i : (x P_i y \rightarrow x P_i' y) \wedge (x I_i y \rightarrow x R_i' y)] \rightarrow [(x P y \rightarrow x P' y) \wedge (x I y \rightarrow x R' y)]$$

Definition 4: Neutrality (N): For all pairs $(R_1, \cdots, R_N)$ and $(R_1', \cdots, R_N')$ of N-tuples of individual orderings in the domain of a social decision rule f, which maps them respectively into R and R', if

$$\forall x,y,z,w \in S : [(\forall i: x R_i y \leftrightarrow z R_i' w) \wedge (\forall i: y R_i x \leftrightarrow w R_i' z)] \rightarrow [(x R y \leftrightarrow z R' w) \wedge (y R x \leftrightarrow w R' z)]$$, then and only then neutrality holds.

Definition 5: Strict Pareto-criterion $(\bar{P})$: $\forall x,y \in S$:

$$[(\forall i : x R_i y \text{ and } \exists i : x P_i y) \rightarrow x P y] \text{ and } (\forall i : x I_i y \rightarrow x I y).$$

Definition 6: R is acyclic over S iff

$$\forall x_1, \cdots, x_j \in S :$$

$$x_1 P x_2 \wedge \cdots \wedge x_{j-1} P x_j \rightarrow x_1 R x_j .$$

Definition 7: R is quasi-transitive over S iff

$$\forall x,y,z \in S :$$

$$x P y \wedge y P z \rightarrow x P z .$$

Definition 8: R is transitive over S iff $\forall x,y,z \in S$: $x R y \wedge y R z \rightarrow x R z$.

Definition 9: A set of individuals V is decisive iff for all $x,y \in S$: $\forall i \in V : x P_i y \rightarrow x P y$.

Definition 10: A set of individuals V is a minimal decisive set iff it is a decisive set and no proper subset of it is a decisive set.

Definition 11: A set of individuals V is (N-A) decisive iff $\forall x,y \in S: \forall i \in A : x I_i y \wedge \forall i \in V : x P_i y \rightarrow x P y$, where $A \subsetneq N$ and $A \cap V = \emptyset$.

Definition 12: A set of individuals V is minimally (N-A) decisive iff V is (N-A)-decisive and no proper subset of V is (N-A)-decisive.

Definition 13: Condition of Unrestricted Domain (U): The domain of f must include all logically possible combinations of individual orderings.

Theorem 1 : Let f satisfy U,I,P,M and N. Then, f yields acyclic R iff for every A, every set of k ($k \leq n$) minimal (N-A)-decisive sets have nonempty intersection, where n is the number of alternatives in S(the set of social alternatives)

Proof: Suppose for some A, a set of k($k \leq n$) minimal (N-A)-decisive sets have empty intersection. Assume the following configuration of preferences.

$$\forall i \in A \quad : \quad x_1 \, I_i \, x_2 \, I_i \cdots I_i \, x_k$$

$$\forall i \in V_1 \quad : \quad x_1 \, P_i \, x_2$$

$$\forall i \in V_2 \quad : \quad x_2 \, P_i \, x_3$$

$$\forall i \in V_{k-1} \quad : \quad x_{k-1} \, P_i \, x_k$$

$$\forall i \in V_k \quad : \quad x_k \, P_i \, x_1$$

Notice that every individual has an ordering over S. This is possible because $\bigcap_{j=1}^{k} V_j$ is empty. Now by the (N-A)-decisiveness of $V_1, \cdots, V_k$ we have

$$x_1 \, P \, x_2, \quad x_2 \, P \, x_3, \quad \cdots, \quad x_{k-1} \, P x_k, \qquad x_k \, P \, x_1 \, .$$

So, acyclicity is violated.

Now assume that acyclicity is violated. Let $\langle R_i \rangle$ violate acyclicity over $(x_1, \cdots, x_k)$. Let the social preference be

$$x_1 \, P \, x_2 \, P \, x_3 \, P \cdots P \, x_k \, P \, x_1 \, .$$

Let N_0 be the set of individuals who are indifferent among all alternatives belonging to $\{x_1, \cdots, x_k\}$. Now construct $\langle R_i' \rangle$ as follows.

(a) $\forall i \in N_0 \; : \{x_1 \, I_i' \cdots I_i' \, x_k\}$

(b) $\forall x, y \in \{x_1, \cdots, x_k\}$:

$(x \, P_i \, y \leftrightarrow x \, P_i' \, y)$, for all $i \in N - N_0$

(c) $\forall\ x_{t_1}, x_{t_2} \in \{x_2, \cdots, x_k\}$:

$x_{t_1}\ I_i\ x_{t_2} \rightarrow x_{t_1}\ P_i^{!}\ x_{t_2}$, iff $t_2 > t_1$, for all $i \in N\text{-}N_o$.

(d) for all $x_t \neq x_1, x_k$

$x_t\ I_i\ x_1 \rightarrow x_1\ P_i^{!}\ x_t$, for all $i \in N\text{-}N_o$

(e) For all $i \in N\text{-}N_o$

$x_1\ I_i\ x_k \rightarrow x_k\ P_i^{!}\ x_1$.

Clearly every individual $\in N$ has either a strong ordering or null ordering over $\{x_1, \cdots, x_k\}$ in the situation $< R_i >$. Also ~~$R = R'$ over $x_1, \cdots, x_k$~~ by M, we have $x_1 P' x_2 P' \cdots P' x_k P' x_1$.

Consider the restriction of $< R_i^{!} >$ over $\{x_1, x_2\}$. We have $\forall\, i \in N_o : x_1\ I_i^{!}\ x_2$, $\forall\, i \in N_1 : x_1\ P_i^{!}\ x_2$ and $\forall\, i \in N\text{-}N_o\text{-}N_1 : x_2\ P_i^{!}\ x_1$. This yields $x_1\ P'\ x_2$. So N_1 is $(N\text{-}N_o)$-decisive by conditions M and N. Hence there exists a minimal $(N\text{-}N_o)$-decisive set $V_1 \subset N_1$. Similarly by considering other pairs of alternatives (x_j, x_{j+1}), the existence of minimal decisive sets $V_2, \cdots, V_k$ can be demonstrated. Suppose the intersection of $\bigcap_{j=1}^{k} V_j$ is nonempty, then for $i \in \bigcap_{j=1}^{k} V_j$ we have

$$x_1\ P_i^{!}\ x_2\ P_i^{!} \cdots P_i^{!}\ x_k\ P_i^{!}\ x_1 .$$

But this violates the assumption that every individual preference relation is an ordering. Hence $\bigcap_{j=1}^{k} V_j$ must be empty. This completes the proof.

In the remainder of this paper we assume $n \geq 3$.

<u>Theorem 2</u> : Let f satisfy U, I, $\bar{P}$, M and N. Then, f yields quasi-transitive R iff for every A, there exists a unique minimal (N-A)-decisive set.

<u>Proof</u> : See Guha [1] for a proof.

<u>Theorem 3</u> : Let f satisfy U, I, $\bar{P}$, M, and N. Then, f yields transitive R iff for every A, there exists a unique minimal (N-A)-decisive set consisting of a single individual.

Sufficiency:

<u>Proof</u> : The uniqueness of (N-A)-minimal decisive set follows from Theorem 2. So the only thing that we have to prove is that the unique minimal (N-A)-decisive set consists of a single individual. Suppose for some A, the unique minimal (N-A) decisive set V consists of more than one individual. Partition V into V_1 and V_2 where V_1 consists of a single individual. Consider the following configuration of individual preferences.

$\forall i \in V_1$	:	x y z
$\forall i \in V_2$	:	z x y
$\forall i \in N-V-A$	:	y z x
$\forall i \in A$	:	(xyz)

As we have $\forall i \in V : x P_i y$ and $\forall i \in A : x I_i y$, we must have x P y, by the (N-A)-decisiveness of V. Now $\forall i \in N-V_2-A : y P_i z$ and $\forall i \in V_2 : z P_i y$ and $\forall i \in A : y I_i z$, so we must have y R z because V_2 is a proper subset of a minimal (N-A)-decisive set.

Now $x P y \wedge y R z \rightarrow x P z$

However only individual 1 prefers x to z. This means that individual 1 is (N-A) decisive. This contradicts the fact that V is a minimal (N-A)-decisive set. This contradiction establishes the sufficiency part.

Necessity part

Assume that for every A, there is a unique minimal (N-A)-decisive set consisting of a single individual. We now show that this implies that R must be transitive. Suppose not, then there exists a situation $\langle R_i \rangle$ which violates transitivity over some triple, say, $\{x,y,z\}$. Without any loss of generality assume

$$y R z \wedge z R x \wedge \sim(yRx)$$

As there is a unique minimal (N-A)-decisive set for every A, R must be quasi-transitive by theorem 2.

In view of quasi-transitivity it follows that R is $x P y \wedge y I z \wedge x I z$. Let the restriction of $\langle R_i \rangle$ over $\{x,y\}$ be as follows: $\forall i \in N_1 : x P_i y$, $\forall i \in N_2 : y P_i x$ and $\forall i \in N_3 : x I_i y$. This yields $x P y$. Hence N_1 is $(N-N_3)$-decisive. Let individual $j \in N_1$ constitute the unique minimal $(N-N_3)$-decisive set. As there exists a unique minimal (N-A)-decisive set consisting of a single individual for every A, it follows that $y I z \rightarrow \forall i \in N : y I_i z$ and $x I z \rightarrow \forall i \in N : x I_i z$. Now $x P_j y$ and $y I_j z \rightarrow x P_j z$ by the transitivity of individual preferences. However this negates $\forall i \in N : x I_i z$, as established above. This contradiction establishes the theorem

In view of above theorems, for the class of functions which satisfy U, I, $\bar{P}$, M, and N we get the following characterization of rationality conditions.

(1) A function violates acyclicity iff for some A, a set of k($k \leq n$) minimal (N-A)-decisive sets have empty intersection.

(2) A function satisfies acyclicity but violates quasi-transitivity iff for some A, there are at least two (N-A)-minimal decisive sets and for every A, every set of k($k \leq n$) minimal (N-A)-decisive sets has nonempty intersection.

(3) A function satisfies quasi-transitivity but violates transitivity iff for every A, there is a unique minimal (N-A)-decisive set and for some A the unique minimal (N-A)-decisive set contains at least two individuals.

(4) A function satisfies transitivity iff for every A, there is a unique minimal (N-A)-decisive set consisting of a single individual.

R E F E R E N C E S

1. Guha, A.S., "Neutrality Monotonicity and the Right of Veto", Econometrica 1972.