The Method of Majority Decision and Rationality Conditions*

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Abstract

This paper extends and simplifies the standard results on the domain conditions for social rationality under the method of majority decision. Inada-type necessary and sufficient conditions for transitivity and quasi-transitivity are derived for every possible case with transitive as well as quasi-transitive individual preferences. A unified approach is adopted for the problem of formulating conditions for social rationality; and for each case a single restriction is formulated as an Inada-type necessary and sufficient condition.

Keywords: Method of Majority Decision, Domain Conditions, Transitivity, Quasi-Transitivity, Acyclicity

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An important problem in the context of social decision rules which do not possess the property of yielding a rational (transitive, quasi-transitive or acyclic) social binary weak preference relation for every profile of individual weak preference relations is that of characterizing sets of binary relations defined over the set of alternatives which are such that if in a profile every individual's weak preference relation belongs to one of these sets then the social weak preference relation generated by the social decision rule is invariably rational. In other words, if $\mathcal{B}$ is a set of binary relations defined over the set of alternatives then the question is that of formulating conditions which would characterize nonempty subsets D of $\mathcal{B}$ which are such that if every individual's binary weak preference relation in a profile belongs to D then the social binary weak preference relation generated by the profile satisfies a particular rationality condition. In the social choice literature pertaining to domain restrictions, a condition which completely characterizes all nonempty subsets D with the property that whenever every individual's binary weak preference relation belongs to D the social binary weak preference relation generated by the social decision rule satisfies a particular rationality condition is called an Inada-type necessary and sufficient condition for that rationality condition. In general, for any social decision rule, the characterizing conditions for a particular rationality condition depend on the number of alternatives, the number of individuals and the set $\mathcal{B}$ whose nonempty subsets are the objects of study. In the context of any social decision rule, the most important of these problems is obviously that of formulating Inada-type necessary and sufficient conditions when $\mathcal{B}$ is the set of all logically possible orderings of the set of alternatives.

In the context of formulating characterizing conditions for social rationality, the method of majority decision has been extensively studied.[1] For the case when $\mathcal{B}$ consists of all logically possible orderings of the set of alternatives, contributions by Black [1948, 1958], Arrow [1963], Inada [1964, 1969], Sen [1966, 1970], Sen and Pattanaik [1969], Pattanaik [1971] and Kelly [1974] have resulted in the establishment of the following propositions: (i) A sufficient condition for transitivity under the method of majority decision (MMD) is that the condition of extremal restriction (ER) holds over every triple of alternatives. If the number of alternatives is at least three and the number of individuals is even and greater than one then the satisfaction of ER over every triple is also an Inada-type necessary condition for transitivity under the MMD. (ii) A sufficient condition for quasi-transitivity under the MMD is that at least one of the three conditions of limited agreement (LA),

[1]For an excellent survey of the literature on domain conditions see Gaertner (2001).

value-restriction (VR) and extremal restriction holds over every triple of alternatives. If the number of social alternatives is at least three and the number of individuals is at least five, then the satisfaction of ER, VR or LA over every triple is also an Inada-type necessary condition for quasi-transitivity under the MMD. In this paper, Inada-type necessary and sufficient conditions are derived for the cases not covered by propositions (i) and (ii). Given that the number of alternatives is at least three, these cases are: Inada-type necessary and sufficient condition for transitivity when the number of individuals is odd and greater than one, for quasi-transitivity when the number of individuals is four, and for quasi-transitivity when the number of individuals is three.

If Inada-type necessary and sufficient conditions are formulated in terms of disjunction of more than one condition then for proving the necessity part one first has to find all different ways in which all the conditions figuring in the disjunction could be violated together. This necessarily makes the proof of the necessity part of a characterization theorem both long and tedious. For this reason in this paper all Inada-type characterization theorems are formulated in terms of a single condition.

By combining propositions (i) and (ii) with the results of this paper, the conditions for quasi-transitivity and transitivity under the MMD, when the number of alternatives is at least three, can be stated as follows: (a) If the number of individuals is even and greater than one then an Inada-type necessary and sufficient condition for transitivity under the MMD is that the condition of extremal restriction holds over every triple of alternatives. (b) If the number of individuals is odd and greater than one then an Inada-type necessary and sufficient condition for transitivity under the MMD is that the condition of weak Latin Square partial agreement (WLSPA) holds over every triple of alternatives. (c) If the number of individuals is at least 5 then an Inada-type necessary and sufficient condition for quasi-transitivity under the MMD is that the condition of Latin Square partial agreement (LSPA) holds over every triple of alternatives. (d) If the number of individuals is four, then an Inada-type necessary and sufficient condition for quasi-transitivity under the MMD is that the condition of weak extremal restriction (WER) holds over every triple of alternatives. (e) If the number of individuals is three, then an Inada-type necessary and sufficient condition for quasi-transitivity under the MMD is that the condition of Latin Square linear ordering restriction (LSLOR) holds over every triple of alternatives. The logical relationships among the five conditions which figure in propositions (a)-(e) are given by: ER implies WLSPA; WLSPA implies LSPA, LSPA itself being logically equivalent to the disjunction of VR, ER and LA; LSPA implies WER; and WER implies LSLOR. It is possible to reformulate extremal restriction in a way that makes it possible to interpret it as a 'partial agreement' condition and the other four conditions as its weakened versions.

In the reformulated version of ER what is required is that if a set of orderings of a triple contains a linear ordering of the triple then in any ordering belonging to the set which is of the same Latin Square as the one associated with the linear ordering, the alternative which is the best in the linear ordering must be considered to be at least as good as the alternative which is the worst in the linear ordering. WLSPA requires fulfilment of ER if a weak Latin Square involving a linear ordering exists. LSPA requires fulfilment of ER in case a Latin Square involving a linear ordering exists. WER requires that in case there is a linear ordering of the triple in question, then it must not be the case that there is an ordering of the triple in which the worst alternative of the linear ordering is uniquely best, there is an ordering of the triple in which the best alternative of the linear ordering is uniquely worst, and both these orderings belong to the same Latin Square which is associated with the linear ordering. LSLOR merely requires that there be no Latin Square involving more than one linear ordering. The formulation of extremal restriction as a 'partial agreement' condition and of the other four conditions as weakened versions of extremal restriction makes it possible to prove Inada-type necessary and sufficient conditions for transitivity and quasi-transitivity for various cases essentially on the basis of a couple of elementary observations elaborated in lemmas 1-4 of the paper, resulting in both considerable simplification of proofs and a unified framework. The simplification which is attained because of the 'Latin Square approach' adopted in this paper is even greater in the case when one deals with reflexive, connected and quasi-transitive binary relations rather than orderings.

For the case when $\mathcal{B}$ consists of all logically possible reflexive, connected and quasi-transitive binary relations over the set of alternatives, contributions by Inada (1970), Fishburn (1970) and Pattanaik (1970) have established that a sufficient condition for quasi-transitivity under the MMD is that at least one of the four conditions of dichotomous preferences (DP), antagonistic preferences (AP), generalized limited agreement (GLA) and generalized value-restriction (GVR) holds over every triple of alternatives. If the number of social alternatives is at least three and the number of individuals is at least five, then the satisfaction of DP, AP, GLA or GVR over every triple is also an Inada-type necessary condition for quasi-transitivity under the MMD. For the remaining cases Inada-type necessary and sufficient conditions are derived in this paper. When the number of alternatives is at least three, the Inada-type necessary and sufficient conditions for quasi-transitivity for the cases of at least 5 individuals, for 4 individuals, for 3 individuals and for 2 individuals are respectively: satisfaction over every triple of Latin Square partial agreement - Q (LSPA-Q)[2], of weak extremal restriction - Q (WER-Q), of Latin

[2]As both LSPA-Q and (DP $\vee$ AP $\vee$ GLA $\vee$ GVR) are Inada-type necessary and sufficient conditions when the number of individuals is at least 5, it follows that they are logically equivalent. In Jain (1986) it is inferred that strict Latin Square Partial agreement (SLSPA) is logically equivalent to (DP $\vee$ AP $\vee$ GLA $\vee$ GVR). Thus, LSPA-Q is logically equivalent to SLSPA. The formulation in terms

Square linear ordering restriction - Q (LSLOR-Q), and of Latin Square intransitive relation restriction - Q (LSIRR-Q). It turns out that the absence of Latin Squares involving intransitive relations is crucial for quasi-transitivity. Indeed, conditions LSPA-Q, WER-Q and LSLOR-Q are merely conjunctions of the requirement that there be no Latin Square involving an intransitive relation with LSPA, WER and LSLOR respectively. Thus the requirements for quasi-transitivity are the same when individual binary weak preference relations are quasi-transitive as when they are transitive except for the added requirement that there be no Latin Square involving an intransitive relation. Needless to say, this way of formulating conditions results in considerable simplification of proofs as well as clearer understanding of domain conditions by making the conditions for the case when individual weak preference relations are quasi-transitive directly comparable to the case when individual weak preference relations are transitive. LSIRR-Q, the weakest of the four conditions merely requires that there be no Latin Square consisting of two intransitive relations or one intransitive relation and one linear ordering.

The paper is divided into five sections. The first section introduces the Latin Square framework within which the Inada-type necessary and sufficient conditions are formulated. The second section deals with the Inada-type necessary and sufficient conditions for transitivity and the third section with conditions for quasi-transitivity, when individual binary weak preference relations are orderings. The fourth section contains characterization theorems for quasi-transitivity when individual binary weak preference relations are reflexive, connected and quasi-transitive. The concluding section is concerned with the question of formulating conditions for acyclicity defined only over triples. The main theorem of this section essentially demonstrates that for any non-trivial set of binary relations $\mathcal{B}$ containing intransitive binary relations no condition defined only over triples can be an Inada-type necessary and sufficient condition for acyclicity.

1 Notation and Definitions

The set of social alternatives and the finite set of individuals constituting the society are denoted by S and L respectively. We assume $\#S = m \geq 3$ and $\#L = n \geq 2$. Each individual $i \in L$ is assumed to have a binary weak preference relation R_i on S. We denote asymmetric parts of binary relations R, R_i, R_j, R^s etc. by P, P_i, P_j, P^s etc. respectively; and symmetric parts by I, I_i, I_j, I^s etc. respectively.

of LSPA-Q, rather than SLSPA, has the advantage of making the condition for the case when individual weak preference relations are quasi-transitive directly comparable with the condition for the case when individual weak preference relations are transitive.

We define a binary relation R on a set S to be (i) reflexive iff $(\forall x \in S)(xRx)$, (ii) connected iff $(\forall x, y \in S)(x \neq y \rightarrow xRy \vee yRx)$, (iii) anti-symmetric iff $(\forall x, y \in S)(xRy \wedge yRx \rightarrow x = y)$, (iv) transitive iff $(\forall x, y, z \in S)(xRy \wedge yRz \rightarrow xRz)$, (v) quasi-transitive iff $(\forall x, y, z \in S)(xPy \wedge yPz \rightarrow xPz)$, (vi) acyclic iff $(\forall x_1, x_2, \ldots, x_n \in S)(x_1Px_2 \wedge x_2Px_3 \wedge \ldots \wedge x_{n-1}Px_n \rightarrow x_1Rx_n)$, where n is a positive integer ≥ 3, (vii) an ordering iff it is reflexive, connected and transitive, and (viii) a linear ordering iff it is reflexive, connected, anti-symmetric and transitive. Throughout this paper it would be assumed that for each $i, i \in L$, R_i is reflexive and connected.

We denote by $\mathcal{C}$ the set of all reflexive and connected binary relations on S, by $\mathcal{A}$ the set of all reflexive, connected and acyclic binary relations on S, by $\mathcal{Q}$ the set of all reflexive, connected and quasi-transitive binary relations on S, and by $\mathcal{T}$ the set of all reflexive, connected and transitive binary relations (orderings) on S. We denote by $N()$ the number of individuals having the preferences specified within the parentheses. The method of majority decision (MMD) f, a function from $\mathcal{G} \subseteq \mathcal{C}^n$ to $\mathcal{C}$, $f : \mathcal{G} \mapsto \mathcal{C}$; is defined by: $(\forall (R_1, ..., R_n) \in \mathcal{G})(\forall x, y \in S)[xRy \leftrightarrow N(xP_iy) \geq N(yP_ix)]$, where $R = f(R_1, \ldots, R_n)$.

Let $A \subseteq S$ and let R be a binary relation on S. We define restriction of R to A, denoted by $R|A$, by $R|A = R \cap (A \times A)$. Throughout this paper D would denote a nonempty set of binary relations defined over the set of alternatives S. We define restriction of D to A, denoted by $D|A$, by $D|A = \{R|A \mid R \in D\}$.

A set of three distinct alternatives will be called a triple of alternatives. Let R be a binary relation on S and let $A = \{x, y, z\} \subseteq S$ be a triple of alternatives. We define $R \in \mathcal{C}$ to be unconcerned over A iff $(\forall a, b \in A)(aIb)$. R is defined to be concerned over A iff it is not unconcerned over A. We denote by n_A the number of individuals who are concerned over the triple A. We define in A, according to R, x to be best iff $(xRy \wedge xRz)$; to be medium iff $[(yRx \wedge xRz) \vee (zRx \wedge xRy)]$; to be worst iff $(yRx \wedge zRx)$; and to be proper medium iff $[(yPx \wedge xRz) \vee (yRx \wedge xPz) \vee (zPx \wedge xRy) \vee (zRx \wedge xPy)]$.

Weak Latin Square (WLS): Let $A = \{x, y, z\} \subseteq S$ be a triple of alternatives and let R^s, R^t, R^u be binary relations on S. The set $\{R^s|A, R^t|A, R^u|A\}$ forms a weak Latin Square over A iff ($\exists$ distinct $a, b, c \in A$) [(in $R^s|A$ a is best and b is medium and c is worst) $\wedge$ (in $R^t|A$ b is best and c is medium and a is worst) $\wedge$ (in $R^u|A$ c is best and a is medium and b is worst)]. The above weak Latin Square will be denoted by $WLS(abca)$.

Remark 1 *$R^s|A, R^t|A, R^u|A$ in the definition of weak Latin Square need not be distinct. $\{xIyIz\}$ forms a weak Latin Square over the triple $\{x, y, z\}$.*

Remark 2 *If $R^s|A, R^t|A, R^u|A$ are orderings over A, then the set $\{R^s|A, R^t|A, R^u|A\}$ forms a weak Latin Square over A iff $(\exists$ distinct $a, b, c \in A)[aR^sbR^sc \wedge bR^tcR^ta \wedge cR^uaR^ub]$.*

Latin Square (LS): Let $A = \{x, y, z\} \subseteq S$ be a triple of alternatives and let R^s, R^t, R^u be binary relations on S. The set $\{R^s|A, R^t|A, R^u|A\}$ forms a Latin Square over A iff ($\exists$ distinct $a, b, c \in A$) [(in $R^s|A$ a is best and b is proper medium and c is worst) $\wedge$ (in $R^t|A$ b is best and c is proper medium and a is worst) $\wedge$ (in $R^u|A$ c is best and a is proper medium and b is worst)]. The above Latin Square will be denoted by $LS(abca)$.

Remark 3 *If $R^s|A, R^t|A, R^u|A$ are orderings over A, then the set $\{R^s|A, R^t|A, R^u|A\}$ forms a Latin Square over A iff $R^s|A, R^t|A, R^u|A$ are concerned over A and $(\exists$ distinct $a, b, c \in A)[aR^sbR^sc \wedge bR^tcR^ta \wedge cR^uaR^ub]$.*

Remark 4 *From the definitions of weak Latin Square and Latin Square, it is clear that if $R^s|A, R^t|A, R^u|A$ are orderings and concerned over A then they form a Latin Square iff they form a weak Latin Square.*

Let $A = \{x, y, z\} \subseteq S$ be a triple of alternatives. For any distinct $a, b, c \in A$, we define:

$T[WLS(abca)] = \{R \in \mathcal{T}|A \mid (aRbRc \vee bRcRa \vee cRaRb)\}$.
$T[LS(abca)] = \{R \in \mathcal{T}|A \mid R$ is concerned over $A \wedge (aRbRc \vee bRcRa \vee cRaRb)\}$.
$Q[LS(abca)] = \{R \in \mathcal{Q}|A \mid$ (a is best and b is proper medium and c is worst in R) $\vee$ (b is best and c is proper medium and a is worst in R) $\vee$ (c is best and a is proper medium and b is worst in R)$\}$.

Thus we have:
$T[WLS(xyzx)] = T[WLS(yzxy)] = T[WLS(zxyz)] = \{xPyPz,\ xPyIz,\ xIyPz,$ $yPzPx,\ yPzIx,\ yIzPx,\ zPxPy,\ zPxIy,\ zIxPy,\ xIyIz\}$
$T[WLS(xzyx)] = T[WLS(zyxz)] = T[WLS(yxzy)] = \{xPzPy,\ xPzIy,\ xIzPy,$ $zPyPx,\ zPyIx,\ zIyPx,\ yPxPz,\ yPxIz,\ yIxPz,\ xIyIz\}$
$T[LS(xyzx)] = T[LS(yzxy)] = T[LS(zxyz)] = T[WLS(xyzx)] - \{xIyIz\}$
$T[LS(xzyx)] = T[LS(zyxz)] = T[LS(yxzy)] = T[WLS(xzyx)] - \{xIyIz\}$
$Q[LS(xyzx)] = Q[LS(yzxy)] = Q[LS(zxyz)] = \{xPyPz,\ xPyIz,\ xIyPz,\ yPzPx,\ yPzIx,$ $yIzPx,\ zPxPy,\ zPxIy,\ zIxPy,\ (xPy, yIz, zIx),\ (yPz, zIx, xIy),\ (zPx, xIy, yIz)\}$
$Q[LS(xzyx)] = Q[LS(zyxz)] = Q[LS(yxzy)] = \{xPzPy,\ xPzIy,\ xIzPy,\ zPyPx,\ zPyIx,$

$zIyPx,\ yPxPz,\ yPxIz,\ yIxPz,\ (yPx, xIz, zIy),\ (xPz, zIy, yIx),\ (zPy, yIx, xIz)\}$.

Now we define several restrictions on sets of orderings.

Extremal Restriction (ER): $D \subseteq \mathcal{T}$ satisfies ER over the triple $A \subseteq S$ iff $(\forall$ distinct $a, b, c \in A)[(\exists R \in D|A)(aPbPc) \rightarrow (\forall R \in D|A \cap T[LS(abca)])(aRc)]$. D satisfies ER iff it satisfies ER over every triple contained in S.[3]
Thus, the satisfaction of extremal restriction by D over the triple A requires that in case $D|A$ contains a linear ordering of A then in every ordering in $D|A$ which belongs to the same Latin Square as the linear ordering, the alternative which is the best in the linear ordering must be at least as good as the alternative which is the worst in the linear ordering.

Weak Latin Square Partial Agreement (WLSPA): $D \subseteq \mathcal{T}$ satisfies WLSPA over the triple $A \subseteq S$ iff $(\forall$ distinct $a, b, c \in A)[(\exists R^s, R^t, R^u \in D|A)(aP^sbP^sc \wedge bR^tcR^ta \wedge cR^uaR^ub) \rightarrow (\forall R \in D|A \cap T[LS(abca)])(aRc)]$. D satisfies WLSPA iff it satisfies WLSPA over every triple contained in S.
The satisfaction of weak Latin Square partial agreement by D over the triple A requires that in case $D|A$ contains a weak Latin Square involving a linear ordering of A then in every ordering in $D|A$ which belongs to the same Latin Square, the alternative which is the best in the linear ordering must be at least as good as the alternative which is the worst in the linear ordering.

Latin Square Partial Agreement (LSPA): $D \subseteq \mathcal{T}$ satisfies LSPA over the triple $A \subseteq S$ iff $(\forall$ distinct $a, b, c \in A)[(\exists R^s, R^t, R^u \in D|A)(R^s, R^t, R^u$ are concerned over $A \wedge aP^sbP^sc \wedge bR^tcR^ta \wedge cR^uaR^ub) \rightarrow (\forall R \in D|A \cap T[LS(abca)])(aRc)]$. D satisfies LSPA iff it satisfies LSPA over every triple contained in S.
The satisfaction of Latin Square partial agreement by D over the triple A requires that in case $D|A$ contains a Latin Square involving a linear ordering of A then in every ordering in $D|A$ which belongs to the same Latin Square, the alternative which is the best in the linear ordering must be at least as good as the alternative which is the worst in the linear ordering.

Weak Extremal Restriction (WER): $D \subseteq \mathcal{T}$ satisfies WER over the triple $A \subseteq S$ iff $\sim (\exists$ distinct $a, b, c \in A)(\exists R^s, R^t, R^u \in D|A)(aP^sbP^sc \wedge bR^tcP^ta \wedge cP^uaR^ub)$. D satisfies WER iff it satisfies WER over every triple contained in S.
The satisfaction of weak extremal restriction by D over the triple A requires that $D|A$

[3] Although the definition of extremal restriction given here is quite different from the usual one, it can easily be checked that the two definitions are equivalent to each other.

must not contain a Latin Square which is such that one of the orderings involved in the formation of the Latin Square is a linear ordering and in the other two orderings the alternative which is the worst in the linear ordering is preferred to the alternative which is the best in the linear ordering.

Latin Square Linear Ordering Restriction (LSLOR): $D \subseteq \mathcal{T}$ satisfies LSLOR over the triple $A \subseteq S$ iff $\sim$ ($\exists$ distinct $a, b, c \in A$)($\exists R^s, R^t, R^u \in D|A$)($R^s, R^t, R^u$ are concerned over A $\wedge$ $aP^sbP^sc \wedge bP^tcP^ta \wedge cR^uaR^ub$). D satisfies LSLOR iff it satisfies LSLOR over every triple contained in S.
The satisfaction of Latin Square linear ordering restriction by D over the triple A requires that $D|A$ must not contain a Latin Square involving more than one linear ordering.

Remark 5 *From the definitions of the five conditions defined above it is clear that ER implies WSLPA; WLSPA implies LSPA; LSPA implies WER; and WER implies LSLOR.*

Next, we define four restrictions on sets of reflexive, connected and quasi-transitive binary relations.

Latin Square Partial Agreement - Q (LSPA-Q): $D \subseteq \mathcal{Q}$ satisfies LSPA-Q over the triple $A \subseteq S$ iff $[[(\exists R^s, R^t, R^u \in D|A)(R^s, R^t, R^u$ form a Latin Square over $A) \rightarrow (R^s, R^t, R^u$ are orderings over $A)] \wedge [(\forall$ distinct $a, b, c \in A)[(\exists R^s, R^t, R^u \in D|A \cap \mathcal{T}|A)(R^s, R^t, R^u$ are concerned over A $\wedge$ aP^sbP^sc $\wedge$ bR^tcR^ta $\wedge$ $cR^uaR^ub) \rightarrow (\forall R \in D|A \cap T[LS(abca)])(aRc)]]]$.
D satisfies LSPA-Q iff it satisfies LSPA-Q over every triple contained in S.
The satisfaction of Latin Square partial agreement - Q by D over the triple A requires that there be no Latin Square contained in $D|A$ involving an intransitive binary relation; and that $D|A \cap \mathcal{T}|A$ satisfy Latin Square partial agreement.

Weak Extremal Restriction - Q (WER-Q): $D \subseteq \mathcal{Q}$ satisfies WER-Q over the triple $A \subseteq S$ iff $[[(\exists R^s, R^t, R^u \in D|A)(R^s, R^t, R^u$ form a Latin Square over $A) \rightarrow (R^s, R^t, R^u$ are orderings over $A)] \wedge \sim (\exists$ distinct $a, b, c \in A)(\exists R^s, R^t, R^u \in D|A \cap \mathcal{T}|A)(aP^sbP^sc \wedge bR^tcP^ta \wedge cP^uaR^ub)]$. D satisfies WER-Q iff it satisfies WER-Q over every triple contained in S.
The satisfaction of weak extremal restriction - Q by D over the triple A requires that there be no Latin Square contained in $D|A$ involving an intransitive binary relation; and that $D|A \cap \mathcal{T}|A$ satisfy weak extremal restriction.

Latin Square Linear Ordering Restriction - Q (LSLOR-Q): $D \subseteq \mathcal{Q}$ satisfies LSLOR-Q over the triple $A \subseteq S$ iff $[[(\exists R^s, R^t, R^u \in D|A)(R^s, R^t, R^u$ form a Latin Square over $A)$ $\rightarrow (R^s, R^t, R^u$ are orderings over $A)] \wedge \sim (\exists$ distinct $a, b, c \in A)(\exists R^s, R^t, R^u \in D|A \cap \mathcal{T}|A)(R^s, R^t, R^u$ are concerned over A $\wedge$ $aP^sbP^sc \wedge bP^tcP^ta \wedge cR^uaR^ub)]$. D satisfies

LSLOR-Q iff it satisfies LSLOR-Q over every triple contained in S.
The satisfaction of Latin Square linear ordering restriction - Q by D over the triple A requires that there be no Latin Square contained in $D|A$ involving an intransitive binary relation; and that $D|A \cap \mathcal{T}|A$ satisfy Latin Square linear ordering restriction.

Latin Square Intransitive Relation Restriction - Q (LSIRR-Q): $D \subseteq \mathcal{Q}$ satisfies LSIRR-Q over the triple $A \subseteq S$ iff $\sim (\exists R^s, R^t \in D|A)[(R^s, R^t \text{ form a Latin Square over } A) \wedge (R^s \text{ is intransitive}) \wedge (R^t \text{ is intransitive} \vee R^t \text{ is a linear ordering})]$. D satisfies LSIRR-Q iff it satisfies LSIRR-Q over every triple contained in S.
The satisfaction of Latin Square intransitive relation restriction - Q by D over the triple A requires that there be no Latin Square contained in $D|A$ consisting of two intransitive relations or consisting of one intransitive relation and one linear ordering.

Remark 6 *It is clear from the definitions of the above four conditions that LSPA-Q implies WER-Q; WER-Q implies LSLOR-Q; and LSLOR-Q implies LSIRR-Q.*

2 Conditions for Transitivity when Individual Binary Weak Preference Relations are Transitive

This section is concerned with characterizing sets of orderings $D \in 2^{\mathcal{T}} - \{\emptyset\}$ which are such that every logically possible $(R_1, \ldots, R_n) \in D^n$ gives rise to transitive social binary weak preference relation under the MMD. If the number of individuals is even and greater than one then extremal restriction constitutes a characterizing condition (Theorem 1); and when the number of individuals is odd and greater than or equal to 3 then weak Latin Square partial agreement constitutes a characterizing condition (Theorem 2). Both the results follow directly from two elementary observations about the MMD elaborated in Lemmas 1 and 2.

Lemma 1 *Let $f : \mathcal{G} \mapsto \mathcal{C}$, $\mathcal{G} \subseteq \mathcal{C}^n$, be the method of majority decision. Let $(R_1, \ldots, R_n) \in \mathcal{G}$ and $R = f(R_1, \ldots, R_n)$. Then we have:*
(i) $(\forall \text{ distinct } x, y \in S)[xRy \rightarrow N(xR_iy) \geq \frac{n}{2}]$
(ii) $(\forall \text{ distinct } x, y \in S)[xPy \rightarrow N(xR_iy) > \frac{n}{2}]$.

Proof: As each $R_i, i \in L$, is connected, it follows that for any distinct $x, y \in S$ we have:
$N(xR_iy) + N(yR_ix) \geq n$ (1)
Now, $xRy \rightarrow N(xR_iy) \geq N(yR_ix)$, and (2)
$xPy \rightarrow N(xR_iy) > N(yR_ix)$ (3)
(i) follows from (1) and (2), and (ii) follows from (1) and (3).

Lemma 2 *Let $f : \mathcal{G} \mapsto \mathcal{C}$, $\mathcal{G} \subseteq \mathcal{T}^n$, be the method of majority decision. Let $(R_1, \ldots, R_n) \in \mathcal{G}$ and $R = f(R_1, \ldots, R_n)$. Let $A = \{x, y, z\} \subseteq S$ be a triple of alternatives and suppose xPy, yRz and zRx. Then we have:*
(i) $(\exists i \in L)[xP_iyP_iz \vee yP_izP_ix \vee zP_ixP_iy]$
(ii)$(\exists i, j, k \in L)[R_i|A, R_j|A, R_k|A \in T[LS(xyzx)] \wedge xP_iy \wedge yP_jz \wedge zP_kx]$.

Proof: $xPy \rightarrow N(xP_iy) > N(yP_ix)$ (1)
$yRz \rightarrow N(yP_iz) \geq N(zP_iy)$ (2)
$zRx \rightarrow N(zP_ix) \geq N(xP_iz)$ (3)
(1), (2) and (3) imply respectively:
$N(zP_ixP_iy) + N(zI_ixP_iy) + N(xP_izP_iy) + N(xP_izI_iy) + N(xP_iyP_iz) > N(zP_iyP_ix) + N(zI_iyP_ix) + N(yP_izP_ix) + N(yP_izI_ix) + N(yP_ixP_iz)$ (4)
$N(xP_iyP_iz) + N(xI_iyP_iz) + N(yP_ixP_iz) + N(yP_ixI_iz) + N(yP_izP_ix) \geq N(xP_izP_iy) + N(xI_izP_iy) + N(zP_ixP_iy) + N(zP_ixI_iy) + N(zP_iyP_ix)$ (5)
$N(yP_izP_ix) + N(yI_izP_ix) + N(zP_iyP_ix) + N(zP_iyI_ix) + N(zP_ixP_iy) \geq N(yP_ixP_iz) + N(yI_ixP_iz) + N(xP_iyP_iz) + N(xP_iyI_iz) + N(xP_izP_iy)$ (6)
Adding (4), (5) and (6), we obtain:
$N(xP_iyP_iz) + N(yP_izP_ix) + N(zP_ixP_iy) > N(xP_izP_iy) + N(zP_iyP_ix) + N(yP_ixP_iz)$ (7)
Adding (7) to (4), (5) and (6) we obtain respectively:
$2N(zP_ixP_iy) + 2N(xP_iyP_iz) + N(zI_ixP_iy) + N(xP_iyI_iz) > 2N(zP_iyP_ix) + 2N(yP_ixP_iz) + N(zI_iyP_ix) + N(yP_ixI_iz)$ (8)
$2N(xP_iyP_iz) + 2N(yP_izP_ix) + N(xI_iyP_iz) + N(yP_izI_ix) > 2N(xP_izP_iy) + 2N(zP_iyP_ix) + N(xI_izP_iy) + N(zP_iyI_ix)$ (9)
$2N(yP_izP_ix) + 2N(zP_ixP_iy) + N(yI_izP_ix) + N(zP_ixI_iy) > 2N(yP_ixP_iz) + 2N(xP_izP_iy) + N(yI_ixP_iz) + N(xP_izI_iy)$ (10)
(7)-(10) imply respectively:
$N(xP_iyP_iz) + N(yP_izP_ix) + N(zP_ixP_iy) > 0$ (11)
$N(zP_ixP_iy) + N(xP_iyP_iz) + N(zI_ixP_iy) + N(xP_iyI_iz) > 0$ (12)
$N(xP_iyP_iz) + N(yP_izP_ix) + N(xI_iyP_iz) + N(yP_izI_ix) > 0$ (13)
$N(yP_izP_ix) + N(zP_ixP_iy) + N(yI_izP_ix) + N(zP_ixI_iy) > 0$ (14)
(11) $\rightarrow (\exists i \in L)(xP_iyP_iz \vee yP_izP_ix \vee zP_ixP_iy)$ (15)
(12) $\rightarrow (\exists i \in L)(R_i|A \in T[LS(xyzx)] \wedge xP_iy)$ (16)
(13) $\rightarrow (\exists j \in L)(R_j|A \in T[LS(xyzx)] \wedge yP_jz)$ (17)
(14) $\rightarrow (\exists k \in L)(R_k|A \in T[LS(xyzx)] \wedge zP_kx)$ (18)
(15)-(18) establish the lemma.

Theorem 1 *Let $\#S \geq 3$ and $\#L = n = 2k, k \geq 1$. Let $D \subseteq \mathcal{T}$. Then the method of majority decision f yields transitive social $R, R = f(R_1, \ldots, R_n)$, for every $(R_1, \ldots, R_n) \in D^n$ iff D satisfies the condition of extremal restriction.*

Proof: Suppose f does not yield transitive social R for every $(R_1, \ldots, R_n) \in D^n$. Then $(\exists (R_1, \ldots, R_n) \in D^n)(\exists x, y, z \in S)(xPy \wedge yRz \wedge zRx)$. This, by lemma 2, implies that:
$(\exists i \in L)(xP_iyP_iz \vee yP_izP_ix \vee zP_ixP_iy)$, and (1)
$(\exists i, j, k \in L)[R_i|\{x, y, z\}, R_j|\{x, y, z\}, R_k|\{x, y, z\} \in T[LS(xyzx)] \wedge xP_iy \wedge yP_jz$
$\wedge\, zP_kx]$ (2)
(1) and (2) imply that D violates ER, which establishes the sufficiency of ER.

Let $D \subseteq \mathcal{T}$ violate ER. This implies that $(\exists x, y, z \in S)(\exists R^s, R^t \in D)[xP^syP^sz \wedge (yR^tzP^tx \vee zP^txR^ty)]$. Now consider any $(R_1, \ldots, R_n) \in D^n$ such that $\#\{i \in L \mid R_i = R^s\} = \#\{i \in L \mid R_i = R^t\} = k = \frac{n}{2}$. The MMD then yields $(xIy \wedge yPz \wedge xIz)$ or $(xPy \wedge yIz \wedge xIz)$ depending on whether $R^t|\{x, y, z\}$ is yR^tzP^tx or zP^txR^ty. In either case transitivity is violated, which establishes the theorem.

Theorem 2 *Let $\#S \geq 3$ and $\#L = n = 2k + 1, k \geq 1$. Let $D \subseteq \mathcal{T}$. Then the method of majority decision f yields transitive social $R, R = f(R_1, \ldots, R_n)$, for every $(R_1, \ldots, R_n) \in D^n$ iff D satisfies the condition of weak Latin Square partial agreement.*

Proof: Suppose f does not yield transitive social R for every $(R_1, \ldots, R_n) \in D^n$. Then $(\exists (R_1, \ldots, R_n) \in D^n)(\exists x, y, z \in S)(xPy \wedge yRz \wedge zRx)$. xPy, yRz and zRx imply, by lemma 1, respectively;
$N(xR_iy) > \frac{n}{2}$ (1)
$N(yR_iz) \geq \frac{n}{2}$ (2)
$N(zR_ix) \geq \frac{n}{2}$ (3)
As n is odd, (2) and (3) imply respectively,
$N(yR_iz) > \frac{n}{2}$ (4)
$N(zR_ix) > \frac{n}{2}$ (5)
$(1) \wedge (4) \rightarrow (\exists i \in L)(xR_iyR_iz)$ (6)
$(4) \wedge (5) \rightarrow (\exists i \in L)(yR_izR_ix)$ (7)
$(5) \wedge (1) \rightarrow (\exists i \in L)(zR_ixR_iy)$ (8)
By lemma 2 we have: $(\exists i \in L)(xP_iyP_iz \vee yP_izP_ix \vee zP_ixP_iy)$, and (9)
$(\exists i, j, k \in L)[R_i|\{x, y, z\}, R_j|\{x, y, z\}, R_k|\{x, y, z\} \in T[LS(xyzx)] \wedge xP_iy \wedge yP_jz$
$\wedge\, zP_kx]$ (10)
(6)-(10) imply that WLSPA is violated, which establishes the sufficiency of WLSPA.

Now, let $D \subseteq \mathcal{T}$ violate WLSPA. This implies that $(\exists x, y, z \in S)(\exists R^s, R^t, R^u \in D)[(xP^syP^sz \wedge yR^tzP^tx \wedge zR^uxR^uy) \vee (xP^syP^sz \wedge yR^uzR^ux \wedge zP^txR^ty)]$. Consider any $(R_1, \ldots, R_n) \in D^n$ such that $\#\{i \in L \mid R_i = R^s\} = \#\{i \in L \mid R_i = R^t\} = k = \frac{n-1}{2}$, and $\#\{i \in L \mid R_i = R^u\} = 1$. The MMD then yields $[xRy \wedge yRz \wedge zRx \wedge \sim (yRx \wedge xRz \wedge zRy)]$ violating transitivity, which establishes the theorem.

3 Conditions for Quasi-Transitivity when Individual Binary Weak Preference Relations are Transitive

This section is concerned with characterizing sets of orderings $\mathrm{D} \in 2^{\mathcal{T}} - \{\emptyset\}$ which are such that every logically possible $(R_1, \ldots, R_n) \in D^n$ gives rise to quasi-transitive social binary weak preference relation under the MMD. If the number of individuals is greater than or equal to 5 then Latin Square partial agreement constitutes a characterizing condition (Theorem 3); if the number of individuals is 4 then weak extremal restriction constitutes a characterizing condition (Theorem 4); and if the number of individuals is 3 then Latin Square linear ordering restriction constitutes a characterizing condition (Theorem 5). All the three theorems follow directly from Lemmas 2 and 4. Lemma 4 is a simple consequence of Lemma 3 which is similar to Lemma 1.

Lemma 3 *Let $f : \mathcal{G} \mapsto \mathcal{C}$, $\mathcal{G} \subseteq \mathcal{C}^n$, be the method of majority decision. Let $(R_1, \ldots, R_n) \in \mathcal{G}$ and $R = f(R_1, \ldots, R_n)$. Then we have:*
(i) $(\forall$ *distinct* $x, y, z \in S)[xRy \rightarrow N(R_i$ *concerned over* $A = \{x, y, z\} \wedge xR_iy) \geq \frac{n_A}{2}]$
(ii)$(\forall$ *distinct* $x, y, z \in S)[xPy \rightarrow N(R_i$ *concerned over* $A = \{x, y, z\} \wedge xR_iy) > \frac{n_A}{2}]$.

Proof: Let $A = \{x, y, z\} \subseteq S$ be a triple of alternatives.
$xRy \rightarrow N(xP_iy) \geq N(yP_ix)$, and (1)
$xPy \rightarrow N(xP_iy) > N(yP_ix)$ (2)
Adding $N(R_i$ concerned over $A \wedge xI_iy)$ to both sides of inequalities of (1) and (2) we obtain:
$xRy \rightarrow N(R_i$ concerned over $A \wedge xR_iy) \geq N(R_i$ concerned over $A \wedge yR_ix)$, (3)
and
$xPy \rightarrow N(R_i$ concerned over $A \wedge xR_iy) > N(R_i$ concerned over $A \wedge yR_ix)$ (4)
As each $R_i, i \in L$, is connected, it follows that: $N(R_i$ concerned over $A \wedge xR_iy) + N(R_i$ concerned over $A \wedge yR_ix) \geq n_A$ (5)
(i) follows from (3) and (5); and (ii) follows from (4) and (5).

Lemma 4 *Let $f : \mathcal{G} \mapsto \mathcal{C}$, $\mathcal{G} \subseteq \mathcal{T}^n$, be the method of majority decision. Let $(R_1, \ldots, R_n) \in \mathcal{G}$ and $R = f(R_1, \ldots, R_n)$. Let $A = \{x, y, z\} \subseteq S$ be a triple of alternatives and suppose xPy, yPz and zRx. Then we must have:* $(\exists i, j, k \in L)[R_i, R_j, R_k$ *are concerned over* $\{x, y, z\} \wedge xR_iyR_iz \wedge yR_jzR_jx \wedge zR_kxR_ky]$.

Proof: By Lemma 3, xPy, yPz and zRx imply respectively:
$N(R_i$ concerned over $A \wedge xR_iy) > \frac{n_A}{2}$ (1)
$N(R_i$ concerned over $A \wedge yR_iz) > \frac{n_A}{2}$ (2)
$N(R_i$ concerned over $A \wedge zR_ix) \geq \frac{n_A}{2}$ (3)

$(1) \wedge (2) \rightarrow (\exists i \in L)(R_i \text{ concerned over } A \wedge xR_iyR_iz)$ (4)

$(2) \wedge (3) \rightarrow (\exists j \in L)(R_j \text{ concerned over } A \wedge yR_jzR_jx)$ (5)

$(3) \wedge (1) \rightarrow (\exists k \in L)(R_k \text{ concerned over } A \wedge zR_kxR_ky)$ (6)

(4)-(6) establish the lemma.

Theorem 3 *Let $\#S \geq 3$ and $\#L = n \geq 5$. Let $D \subseteq \mathcal{T}$. Then the method of majority decision f yields quasi-transitive social $R, R = f(R_1, \ldots, R_n)$, for every $(R_1, \ldots, R_n) \in D^n$ iff D satisfies the condition of Latin Square partial agreement.*

Proof: Suppose f does not yield quasi-transitive social R for every $(R_1, \ldots, R_n) \in D^n$. Then $(\exists(R_1, \ldots, R_n) \in D^n)(\exists x, y, z \in S)(xPy \wedge yPz \wedge zRx)$. By Lemmas 4 and 2 we obtain:

$(\exists i, j, k \in L)(R_i, R_j, R_k$ are concerned over $\{x, y, z\} \wedge xR_iyR_iz \wedge yR_jzR_jx$
$\wedge\, zR_kxR_ky)$ (1)

$(\exists i \in L)(xP_iyP_iz \vee yP_izP_ix \vee zP_ixP_iy)$ (2)

$(\exists i, j, k \in L)[R_i|\{x, y, z\},\ R_j|\{x, y, z\},\ R_k|\{x, y, z\} \in T[LS(xyzx)] \wedge xP_iy \wedge yP_jz \wedge zP_kx]$ (3)

(1)-(3) imply that LSPA is violated, which establishes the sufficiency of LSPA.

Now, let $D \subseteq \mathcal{T}$ violate LSPA. This implies that $(\exists x, y, z \in S)(\exists R^s, R^t, R^u \in D)[(xP^syP^sz \wedge yR^tzP^tx \wedge zP^uxR^uy) \vee (xP^tyP^tz \wedge yR^szP^sx \wedge zR^uxP^uy) \vee (xP^tyP^tz \wedge yP^uzR^ux \wedge zP^sxR^sy)]$. If $n = 3k, k \geq 2$, consider any $(R_1, \ldots, R_n) \in D^n$ such that $\#\{i \in L \mid R_i = R^s\} = k+1, \#\{i \in L \mid R_i = R^t\} = k$ and $\#\{i \in L \mid R_i = R^u\} = k-1$; if $n = 3k+1, k \geq 2$, consider any $(R_1, \ldots, R_n) \in D^n$ such that $\#\{i \in L \mid R_i = R^s\} = k+1, \#\{i \in L \mid R_i = R^t\} = k$ and $\#\{i \in L \mid R_i = R^u\} = k$; and if $n = 3k+2, k \geq 1$, consider any $(R_1, \ldots, R_n) \in D^n$ such that $\#\{i \in L \mid R_i = R^s\} = k+1, \#\{i \in L \mid R_i = R^t\} = k+1$ and $\#\{i \in L \mid R_i = R^u\} = k$. In each case MMD yields social R violating quasi-transitivity over $\{x, y, z\}$, which establishes the theorem.

Theorem 4 *Let $\#S \geq 3$ and $\#L = n = 4$. Let $D \subseteq \mathcal{T}$. Then the method of majority decision f yields quasi-transitive social $R, R = f(R_1, \ldots, R_4)$, for every $(R_1, \ldots, R_4) \in D^4$ iff D satisfies the condition of weak extremal restriction.*

Proof: Suppose f does not yield quasi-transitive social R for every $(R_1, \ldots, R_4) \in D^4$. Then $(\exists(R_1, \ldots, R_4) \in D^4)(\exists x, y, z \in S)(xPy \wedge yPz \wedge zRx)$. By Lemmas 4 and 2 we obtain:

$(\exists i, j, k \in L)(R_i, R_j, R_k$ are concerned over $\{x, y, z\} \wedge xR_iyR_iz \wedge yR_jzR_jx$
$\wedge\, zR_kxR_ky)$ (1)

$(\exists i \in L)(xP_iyP_iz \vee yP_izP_ix \vee zP_ixP_iy)$ (2)

$(1) \rightarrow (\exists i, j, k \in L)(xP_iz \wedge yP_jx \wedge zP_ky)$ (3)

$(\exists j \in L)(yP_jx) \wedge xPy \rightarrow N(xP_iy) \geq 2 \wedge N(xR_iy) = 3 \wedge N(yP_ix) = 1$ (4)
$(4) \rightarrow N(R_i$ concerned over $\{x, y, z\} \wedge yR_izR_ix) = 1 \wedge N(zP_iyP_ix) = 0$
$\wedge N(yP_ixP_iz) = 0$ (5)
$(\exists k \in L)(zP_ky) \wedge yPz \rightarrow N(yP_iz) \geq 2 \wedge N(yR_iz) = 3 \wedge N(zP_iy) = 1$ (6)
$(6) \rightarrow N(R_i$ concerned over $\{x, y, z\} \wedge zR_ixR_iy) = 1 \wedge N(xP_izP_iy) = 0$ (7)
$(1) \wedge (5) \wedge (7) \rightarrow N(xR_iyR_iz) = 2 \wedge N(R_i$ concerned over $\{x, y, z\} \wedge xR_iyR_iz) \geq 1 \wedge N(R_i$ concerned over $\{x, y, z\} \wedge yR_izR_ix) = 1 \wedge N(R_i$ concerned over $\{x, y, z\} \wedge zR_ixR_iy) = 1$ (8)
$zRx \wedge N(xP_iz) = 1 \wedge (4) \wedge (6) \wedge (8) \rightarrow (\exists i, j, k \in L)[(xP_iyP_iz \wedge yP_izP_ix \wedge zR_ixP_iy) \vee (xP_iyP_iz \wedge yP_izR_ix \wedge zP_ixP_iy)]$
$\rightarrow$ WER is violated. (9)
$zRx \wedge N(xP_iz) = 2 \wedge (4) \wedge (6) \wedge (8) \rightarrow (\exists i, j, k \in L)(xP_iyP_iz \wedge yR_jzP_jx \wedge zP_kxR_ky) \vee (\exists i, j, k, l \in L)(xP_iyI_iz \wedge xI_jyP_jz \wedge yP_kzP_kx \wedge zP_lxP_ly)$
$\rightarrow$ WER is violated. (10)
(9) and (10) establish that WER is violated, which proves sufficiency of WER.

Suppose $D \subseteq \mathcal{T}$ violates WER. This implies that $(\exists x, y, z \in S)(\exists R^s, R^t, R^u \in D)(xP^syP^sz \wedge yR^tzP^tx \wedge zP^uxR^uy)$. Consider any $(R_1, \dots, R_4) \in D^4$ such that $\#\{i \in L \mid R_i = R^s\} = 2$, $\#\{i \in L \mid R_i = R^t\} = 1$ and $\#\{i \in L \mid R_i = R^u\} = 1$. MMD then yields $(xPy \wedge yPz \wedge zIx)$, which violates quasi-transitivity. This establishes the theorem.

Theorem 5 *Let $\#S \geq 3$ and $\#L = n = 3$. Let $D \subseteq \mathcal{T}$. Then the method of majority decision f yields quasi-transitive social $R, R = f(R_1, R_2, R_3)$, for every $(R_1, R_2, R_3) \in D^3$ iff D satisfies the condition of Latin Square linear ordering restriction.*

Proof: Suppose f does not yield quasi-transitive social R for every $(R_1, R_2, R_3) \in D^3$. Then $(\exists (R_1, R_2, R_3) \in D^3)(\exists x, y, z \in S)(xPy \wedge yPz \wedge zRx)$. By lemma 4 we have:
$(\exists i, j, k \in L)(R_i, R_j, R_k$ are concerned over $\{x, y, z\} \wedge xR_iyR_iz \wedge yR_jzR_jx$
$\wedge zR_kxR_ky)$ (1)
$\rightarrow (\exists i, j, k \in L)(xP_iz \wedge yP_jx \wedge zP_ky)$ (2)
$xPy \wedge (\exists j \in L)(yP_jx) \wedge (1) \rightarrow (\exists i, j, k \in L)(R_i, R_j, R_k$ are concerned over $\{x, y, z\} \wedge xP_iyR_iz \wedge yR_jzR_jx \wedge zR_kxP_ky)$ (3)
$yPz \wedge (\exists k \in L)(zP_ky) \wedge (3) \rightarrow (\exists i, j, k \in L)(xP_iyP_iz \wedge yP_jzR_jx \wedge zR_kxP_ky)$ (4)
$zRx \wedge (\exists i \in L)(xP_iz) \wedge (4) \rightarrow (\exists i, j, k \in L)[(xP_iyP_iz \wedge yP_jzP_jx \wedge zR_kxP_ky) \vee (xP_iyP_iz \wedge yP_jzR_jx \wedge zP_kxP_ky)]$ (5)
(5) implies that LSLOR is violated, which establishes sufficiency of LSLOR.

Suppose $D \subseteq \mathcal{T}$ violates LSLOR. This implies that $(\exists x, y, z \in S)(\exists R^s, R^t, R^u \in D)[xP^syP^sz \wedge yP^tzP^tx \wedge (zP^uxR^uy \vee zR^uxP^uy)]$. Consider any $(R_1, R_2, R_3) \in D^3$ such that $\#\{i \in L \mid$

$R_i = R^s\} = \#\{i \in L \mid R_i = R^t\} = \#\{i \in L \mid R_i = R^u\} = 1$. MMD then yields $(xRy \wedge yPz \wedge zPx)$ or $(xPy \wedge yPz \wedge zRx)$ depending on whether R^u over $\{x, y, z\}$ is $zP^u xR^u y$ or $zR^u xP^u y$. As quasi-transitivity is violated in either case the theorem is established.

4 Conditions for Quasi-Transitivity when Individual Binary Weak Preference Relations are Quasi-Transitive

In this section sets of reflexive, connected and quasi-transitive binary relations $D \in 2^{\mathcal{Q}} - \{\emptyset\}$ which are such that every logically possible $(R_1, \ldots, R_n) \in D^n$ gives rise to quasi-transitive social binary weak preference relation under the MMD are characterized. If the number of individuals is greater than or equal to 5 then Latin Square partial agreement - Q constitutes a characterizing condition (Theorem 6); if the number of individuals is 4 then weak extremal restriction - Q constitutes a characterizing condition (Theorem 7); if the number of individuals is 3 then Latin Square linear ordering restriction - Q constitutes a characterizing condition (Theorem 8); and if the number of individuals is 2 then Latin Square intransitive relation restriction - Q constitutes a characterizing condition (Theorem 9). The theorems essentially follow from Lemma 5 which is the counterpart of conjunction of Lemmas 2 and 4 for the case when individual binary weak preference relations are quasi-transitive.

Lemma 5 *Let $f : \mathcal{G} \mapsto \mathcal{C}$, $\mathcal{G} \subseteq \mathcal{Q}^n$, be the method of majority decision. Let $(R_1, \ldots, R_n) \in \mathcal{G}$ and $R = f(R_1, \ldots, R_n)$. Let $A = \{x, y, z\} \subseteq S$ be a triple of alternatives and suppose xPy, yPz and zRx. Then we have:*
(i)$(\exists i, j, k \in L)[R_i|A, R_j|A, R_k|A$ form Latin Square LS(xyzx) over $A]$
(ii)$(\exists i \in L)[xP_iyP_iz \vee yP_izP_ix \vee zP_ixP_iy \vee (xP_iy \wedge yI_iz \wedge xI_iz) \vee (yP_iz \wedge zI_ix \wedge yI_ix) \vee (zP_ix \wedge xI_iy \wedge zI_iy)]$
(iii)$(\exists i, j, k \in L)[R_i|A, R_j|A, R_k|A \in Q[LS(xyzx)] \wedge (xP_iy \vee R_i|A$ is intransitive$) \wedge (yP_jz \vee R_j|A$ is intransitive$) \wedge (zP_kx \vee R_k|A$ is intransitive$)]$.

Proof: $xPy \rightarrow N(xP_iy) > N(yP_ix)$ (1)
$yPz \rightarrow N(yP_iz) > N(zP_iy)$ (2)
$zRx \rightarrow N(zP_ix) \geq N(xP_iz)$ (3)

(1), (2) and (3) imply respectively:
$N(zP_ixP_iy) + N(zI_ixP_iy) + N(xP_izP_iy) + N(xP_izI_iy) + N(xP_iyP_iz) + N(xP_iy \wedge yI_iz \wedge xI_iz) > N(zP_iyP_ix) + N(zI_iyP_ix) + N(yP_izP_ix) + N(yP_izI_ix) + N(yP_ixP_iz) + N(yP_ix \wedge xI_iz \wedge yI_iz)$ (4)

$N(xP_iyP_iz) + N(xI_iyP_iz) + N(yP_ixP_iz) + N(yP_ixI_iz) + N(yP_izP_ix) + N(yP_iz \wedge zI_ix \wedge yI_ix) > N(xP_izP_iy) + N(xI_izP_iy) + N(zP_ixP_iy) + N(zP_ixI_iy) + N(zP_iyP_ix) + N(zP_iy \wedge yI_ix \wedge zI_ix)$ (5)

$N(yP_izP_ix) + N(yI_izP_ix) + N(zP_iyP_ix) + N(zP_iyI_ix) + N(zP_ixP_iy) + N(zP_ix \wedge xI_iy \wedge zI_iy) \geq N(yP_ixP_iz) + N(yI_ixP_iz) + N(xP_iyP_iz) + N(xP_iyI_iz) + N(xP_izP_iy) + N(xP_iz \wedge zI_iy \wedge xI_iy)$ (6)

By adding (4) and (5) we obtain:

$2N(xP_iyP_iz) + N(xP_iyI_iz) + N(xI_iyP_iz) + N(xP_iy \wedge yI_iz \wedge xI_iz) + N(yP_iz \wedge zI_ix \wedge yI_ix) > 2N(zP_iyP_ix) + N(zP_iyI_ix) + N(zI_iyP_ix) + N(zP_iy \wedge yI_ix \wedge zI_ix) + N(yP_ix \wedge xI_iz \wedge yI_iz)$ (7)

(7) $\rightarrow N(xP_iyP_iz) + N(xP_iyI_iz) + N(xI_iyP_iz) + N(xP_iy \wedge yI_iz \wedge xI_iz) + N(yP_iz \wedge zI_ix \wedge yI_ix) > 0$ (8)

(8) $\rightarrow (\exists i \in L)[$in $R_i|A$ x is best $\wedge$ y is proper medium $\wedge$ z is worst$]$ (9)

Analogously we can show that:

(5) $\wedge$ (6) $\rightarrow (\exists j \in L)[$in $R_j|A$ y is best $\wedge$ z is proper medium $\wedge$ x is worst$]$ (10)

(6) $\wedge$ (4) $\rightarrow (\exists k \in L)[$in $R_k|A$ z is best $\wedge$ x is proper medium $\wedge$ y is worst$]$ (11)

(9), (10) and (11) imply:

$(\exists i, j, k \in L)[R_i|A, R_j|A, R_k|A$ form Latin Square $LS(xyzx)$ over $A]$ (12)

Adding (4), (5) and (6), we obtain:

$N(xP_iyP_iz) + N(yP_izP_ix) + N(zP_ixP_iy) + N(xP_iy \wedge yI_iz \wedge xI_iz) + N(yP_iz \wedge zI_ix \wedge yI_ix) + N(zP_ix \wedge xI_iy \wedge zI_iy) > N(zP_iyP_ix) + N(yP_ixP_iz) + N(xP_izP_iy) + N(zP_iy \wedge yI_ix \wedge zI_ix) + N(yP_ix \wedge xI_iz \wedge yI_iz) + N(xP_iz \wedge zI_iy \wedge xI_iy)$ (13)

(13) $\rightarrow N(xP_iyP_iz) + N(yP_izP_ix) + N(zP_ixP_iy) + N(xP_iy \wedge yI_iz \wedge xI_iz) + N(yP_iz \wedge zI_ix \wedge yI_ix) + N(zP_ix \wedge xI_iy \wedge zI_iy) > 0$ (14)

(14) $\rightarrow (\exists i \in L)[xP_iyP_iz \vee yP_izP_ix \vee zP_ixP_iy \vee (xP_iy \wedge yI_iz \wedge xI_iz) \vee (yP_iz \wedge zI_ix \wedge yI_ix) \vee (zP_ix \wedge xI_iy \wedge zI_iy)]$ (15)

Adding (4) and (13)we obtain:

$2N(zP_ixP_iy) + N(zI_ixP_iy) + 2N(xP_iyP_iz) + N(xP_iyI_iz) + 2N(xP_iy \wedge yI_iz \wedge xI_iz) + N(yP_iz \wedge zI_ix \wedge yI_ix) + N(zP_ix \wedge xI_iy \wedge zI_iy) > 2N(zP_iyP_ix) + N(zI_iyP_ix) + 2N(yP_ixP_iz) + N(yP_ixI_iz) + 2N(yP_ix \wedge xI_iz \wedge yI_iz) + N(xP_iz \wedge zI_iy \wedge xI_iy) + N(zP_iy \wedge yI_ix \wedge zI_ix)$ (16)

(16) $\rightarrow (\exists i \in L)[R_i|A \in Q[LS(xyzx)] \wedge (xP_iy \vee R_i|A$ is intransitive$)]$ (17)

Analogously we can show that:

(5) $\wedge$ (13) $\rightarrow (\exists j \in L)[R_j|A \in Q[LS(xyzx)] \wedge (yP_jz \vee R_j|A$ is intransitive$)]$ (18)

$(6) \wedge (13) \to (\exists k \in L)[R_k|A \in Q[LS(xyzx)] \wedge (zP_kx \vee R_k|A \text{ is intransitive})]$ (19)

(17)-(19) imply:

$(\exists i, j, k \in L)[R_i|A, R_j|A, R_k|A \in Q[LS(xyzx)] \wedge (xP_iy \vee R_i|A \text{ is intransitive}) \wedge (yP_jz \vee R_j|A \text{ is intransitive}) \wedge (zP_kx \vee R_k|A \text{ is intransitive})]$ (20)

(12), (15) and (20) establish the lemma.

Theorem 6 *Let $\#S \geq 3$ and $\#L = n \geq 5$. Let $D \subseteq \mathcal{Q}$. Then the method of majority decision f yields quasi-transitive social R, $R = f(R_1, \ldots, R_n)$, for every $(R_1, \ldots, R_n) \in D^n$ iff D satisfies the condition of Latin Square partial agreement - Q.*

Proof: Suppose f does not yield quasi-transitive binary relation R for every $(R_1, \ldots, R_n) \in D^n, D \subseteq \mathcal{Q}$. Then $(\exists (R_1, \ldots, R_n) \in D^n)(\exists x, y, z \in S)(xPy \wedge yPz \wedge zRx)$. Denote $\{x, y, z\}$ by A. By lemma 5 we obtain:

$(i)(\exists i, j, k \in L)[R_i|A, R_j|A, R_k|A$ form Latin Square $LS(xyzx)$ over $A]$

$(ii)(\exists i \in L)[xP_iyP_iz \vee yP_izP_ix \vee zP_ixP_iy \vee (xP_iy \wedge yI_iz \wedge xI_iz) \vee (yP_iz \wedge zI_ix \wedge yI_ix) \vee (zP_ix \wedge xI_iy \wedge zI_iy)]$

$(iii)(\exists i, j, k \in L)[R_i|A, R_j|A, R_k|A \in Q[LS(xyzx)] \wedge (xP_iy \vee R_i|A \text{ is intransitive}) \wedge (yP_jz \vee R_j|A \text{ is intransitive}) \wedge (zP_kx \vee R_k|A \text{ is intransitive})]$.

If $(\exists i \in L)[R_i|A \in Q[LS(xyzx)] \wedge R_i|A$ is intransitive$]$ then in view of (i) there is a Latin Square involving an intransitive binary relation; which would imply violation of LSPA-Q. (1)

If $\sim (\exists i \in L)[R_i|A \in Q[LS(xyzx)] \wedge R_i|A$ is intransitive$]$ then (i)-(iii) imply that: there exist $R_i|A, R_j|A, R_k|A \in T[LS(xyzx)]$, of which at least one is a linear ordering over A, which form a Latin Square; and furthermore $(\exists i, j, k \in L)[R_i|A, R_j|A, R_k|A \in T[LS(xyzx)] \wedge xP_iy \wedge yP_jz \wedge zP_kx]$. This implies that LSPA-Q is violated. (2)

(1) and (2) establish the sufficiency of LSPA-Q.

Suppose $D \subseteq \mathcal{Q}$ violates LSPA-Q. Then there is some triple $A = \{x, y, z\}$ over which LSPA-Q is violated. Violation of LSPA-Q over the triple A implies that there exists a Latin Square over A involving an intransitive binary relation or LSPA is violated over A. If LSPA is violated over A then by Theorem 3 there exists $(R_1, \ldots, R_n) \in D^n$ for which $R = f(R_1, \ldots, R_n)$ violates quasi-transitivity. If there exists a Latin Square over A involving an intransitive binary relation then we must have: $(\exists \text{ distinct } a, b, c \in \{x, y, z\})(\exists R^s, R^t \in D)[[(aP^sb \wedge bI^sc \wedge aI^sc) \wedge (bP^tc \wedge cI^ta \wedge bI^ta)] \vee [(aP^sb \wedge bI^sc \wedge aI^sc) \wedge (bR^tcR^ta \wedge R^t|A$

is concerned)]]. Consider any $(R_1, \ldots, R_n) \in D^n$ such that $\#\{i \in L \mid R_i = R^s\} = n-1, \#\{i \in L \mid R_i = R^t\} = 1$. Then in each case MMD yields an R which violates quasi-transitivity.

Theorem 7 *Let $\#S \geq 3$ and $\#L = n = 4$. Let $D \subseteq \mathcal{Q}$. Then the method of majority decision f yields quasi-transitive social $R, R = f(R_1, \ldots, R_4)$, for every $(R_1, \ldots, R_4) \in D^4$ iff D satisfies the condition of weak extremal restriction - Q.*

Proof: Suppose f does not yield quasi-transitive social R for every $(R_1, \ldots, R_4) \in D^4, D \subseteq \mathcal{Q}$. Then $(\exists (R_1, \ldots, R_4) \in D^4)(\exists x, y, z \in S)(xPy \wedge yPz \wedge zRx)$. By lemma 5 we obtain:
$(\exists i, j, k \in L)[R_i|A, R_j|A, R_k|A$ form Latin Square $LS(xyzx)$ over $A]$ (1)
$(\exists i \in L)[xP_iyP_iz \vee yP_izP_ix \vee zP_ixP_iy \vee (xP_iy \wedge yI_iz \wedge xI_iz) \vee (yP_iz \wedge zI_ix \wedge yI_ix) \vee (zP_ix \wedge xI_iy \wedge zI_iy)]$ (2)

If $(\exists i \in L)$ $[R_i|A \in Q[LS(xyzx)] \wedge R_i|A$ is intransitive] then in view of (1) there is a Latin Square involving an intransitive binary relation; which would imply violation of WER-Q. (3)

If $\sim (\exists i \in L)[R_i|A \in Q[LS(xyzx)] \wedge R_i|A$ is intransitive] then (1) and (2) imply that: there exist $R_i|A, R_j|A, R_k|A \in T[LS(xyzx)]$ which form a Latin Square over A, with at least one of them being a linear ordering. (4)
$(4) \rightarrow (\exists i, j, k \in L)(xP_iz \wedge yP_jx \wedge zP_ky)$ (5)
$(\exists j \in L)(yP_jx) \wedge xPy \rightarrow N(xP_iy) \geq 2 \wedge N(xR_iy) = 3 \wedge N(yP_ix) = 1$ (6)
$(6) \rightarrow N(R_i$ concerned over $\{x, y, z\} \wedge yR_izR_ix) = 1 \wedge N(zP_iyP_ix) = 0 \wedge N(yP_ixP_iz) = 0 \wedge N(yP_ix \wedge xI_iz \wedge yI_iz) = 0$ (7)
$(\exists k \in L)(zP_ky) \wedge yPz \rightarrow N(yP_iz) \geq 2 \wedge N(yR_iz) = 3 \wedge N(zP_iy) = 1$ (8)
$(8) \rightarrow N(R_i$ concerned over $\{x, y, z\} \wedge zR_ixR_iy) = 1 \wedge N(xP_izP_iy) = 0 \wedge N(zP_iy \wedge yI_ix \wedge zI_ix) = 0$ (9)
$(4) \wedge (7) \wedge (9) \rightarrow N(xR_iy \wedge yR_iz \wedge xR_iz) = 2 \wedge N(R_i$ concerned over $\{x, y, z\} \wedge xR_iyR_iz) \geq 1 \wedge N(R_i$ concerned over $\{x, y, z\} \wedge yR_izR_ix) = 1 \wedge N(R_i$ concerned over $\{x, y, z\} \wedge zR_ixR_iy) = 1$ (10)
$zRx \wedge N(xP_iz) = 1 \wedge (6) \wedge (8) \wedge (10) \rightarrow (\exists i, j, k \in L)[(xP_iyP_iz \wedge yP_izP_ix \wedge zR_ixP_iy) \vee (xP_iyP_iz \wedge yP_izR_ix \wedge zP_ixP_iy)]$
$\rightarrow$ WER-Q is violated. (11)
$zRx \wedge N(xP_iz) = 2 \wedge (\exists i \in L)(xP_iz \wedge zI_iy \wedge xI_iy) \wedge (6) \wedge (8) \wedge (10) \rightarrow (\exists i, j, k \in L)(xP_iyP_iz \wedge yP_izP_ix \wedge zP_ixP_iy)$
$\rightarrow$ WER-Q is violated. (12)
$zRx \wedge N(xP_iz) = 2 \wedge \sim (\exists i \in L)(xP_iz \wedge zI_iy \wedge xI_iy) \wedge (6) \wedge (8) \wedge (10) \rightarrow (\exists i, j, k \in$

$L)(xP_iyP_iz \wedge yR_jzP_jx \wedge zP_kxR_ky) \vee (\exists i,j,k,l \in L)(xP_iyI_iz \wedge xI_jyP_jz \wedge yP_kzP_kx \wedge zP_lxP_ly)$
$\rightarrow$ WER-Q is violated. (13)

(3), (11), (12) and (13) establish the sufficiency of WER.

Suppose $D \subseteq \mathcal{Q}$ violates WER-Q. Then there is some triple $A = \{x, y, z\}$ over which WER-Q is violated. Violation of WER-Q over the triple A implies that there exists a Latin Square over A involving an intransitive binary relation or WER is violated over A. If WER is violated over A then by Theorem 4 there exists $(R_1, \ldots, R_n) \in D^n$ for which $R = f(R_1, \ldots, R_n)$ violates quasi-transitivity. If there exists a Latin Square over A involving an intransitive binary relation then we must have: $(\exists$ distinct $a, b, c \in \{x, y, z\})(\exists R^s, R^t \in D)[[(aP^sb \wedge bI^sc \wedge aI^sc) \wedge (bP^tc \wedge cI^ta \wedge bI^ta)] \vee [(aP^sb \wedge bI^sc \wedge aI^sc) \wedge (bR^tcR^ta \wedge R^t|A$ is concerned)]]. Consider any $(R_1, \ldots, R_n) \in D^n$ such that $\#\{i \in L \mid R_i = R^s\} = 3$ and $\#\{i \in L \mid R_i = R^t\} = 1$. Then in each case MMD yields an R which violates quasi-transitivity.

Theorem 8 *Let $\#S \geq 3$ and $\#L = n = 3$. Let $D \subseteq \mathcal{Q}$. Then the method of majority decision f yields quasi-transitive social $R, R = f(R_1, R_2, R_3)$, for every $(R_1, R_2, R_3) \in D^3$ iff D satisfies the condition of Latin Square linear ordering restriction - Q.*

Proof: Suppose f does not yield quasi-transitive social R for every $(R_1, R_2, R_3) \in D^3$. Then $(\exists (R_1, R_2, R_3) \in D^3)(\exists x, y, z \in S)(xPy \wedge yPz \wedge zRx)$. By lemma 5 we have:
$(\exists i, j, k \in L)[R_i|A, R_j|A, R_k|A$ form Latin Square $LS(xyzx)$ over $A]$ (1)
$(\exists i \in L)[xP_iyP_iz \vee yP_izP_ix \vee zP_ixP_iy \vee (xP_iy \wedge yI_iz \wedge xI_iz) \vee (yP_iz \wedge zI_ix \wedge yI_ix) \vee (zP_ix \wedge xI_iy \wedge zI_iy)]$ (2)

If $(\exists i \in L)$ $[R_i|A \in Q[LS(xyzx)]$ $\wedge$ $R_i|A$ is intransitive] then in view of (1) there is a Latin Square involving an intransitive binary relation; which would imply violation of LSLOR-Q. (3)

$\sim$ $(\exists i \in L)$ $[R_i|A \in Q[LS(xyzx)]$ $\wedge$ $R_i|A$ is intransitive] $\wedge$ (1) $\wedge$ $n = 3$ $\rightarrow$ $(\forall i \in L)(R_i|A$ is transitive) (4)
(4) $\wedge$ (1) $\wedge$ (2) $\rightarrow$ $(\forall i \in L)(R_i|A$ is transitive) $\wedge$ $(\exists i, j, k \in L)(R_i|A, R_j|A, R_k|A$ form Latin Square $LS(xyzx)$ over $A)$ $\wedge$ $(\exists i \in L)(xP_iyP_iz \vee yP_izP_ix \vee zP_ixP_iy)$ (5)
Now, by proceeding as in Theorem 5 one can show that (5) implies that LSLOR is violated; which coupled with (3) establishes the sufficiency of LSLOR-Q for quasi-transitivity.

Suppose $D \subseteq \mathcal{Q}$ violates LSLOR-Q. Then there is some triple $A = \{x, y, z\}$ over which LSLOR-Q is violated. Violation of LSLOR-Q over the triple A implies that there exists a Latin Square over A involving an intransitive binary relation or LSLOR is violated over A. If LSLOR is violated over A then by Theorem 5 there exists $(R_1, R_2, R_3) \in D^3$ for which $R = f(R_1, R_2, R_3)$ violates quasi-transitivity. If there exists a Latin Square over A involving an intransitive binary relation then we must have: $(\exists$ distinct $a, b, c \in \{x, y, z\})(\exists R^s, R^t \in D)[[(aP^sb \wedge bI^sc \wedge aI^sc) \wedge (bP^tc \wedge cI^ta \wedge bI^ta)] \vee [(aP^sb \wedge bI^sc \wedge aI^sc) \wedge (bR^tcR^ta \wedge R^t|A$ is concerned$)]]$. Consider any $(R_1, R_2, R_3) \in D^3$ such that $\#\{i \in L \mid R_i = R^s\} = 2$ and $\#\{i \in L \mid R_i = R^t\} = 1$. Then in each case MMD yields an R which violates quasi-transitivity.

Theorem 9 *Let $\#S \geq 3$ and $\#L = n = 2$. Let $D \subseteq \mathcal{Q}$. Then the method of majority decision f yields quasi-transitive social $R, R = f(R_1, R_2)$, for every $(R_1, R_2) \in D^2$ iff D satisfies the condition of Latin Square intransitive relation restriction - Q.*

Proof: Suppose f does not yield quasi-transitive social R for every $(R_1, R_2) \in D^2$. Then $(\exists (R_1, R_2) \in D^2)(\exists x, y, z \in S)(xPy \wedge yPz \wedge zRx)$.

$$xPy \rightarrow N(xP_iy) \geq 1 \wedge N(xR_iy) = 2 \quad (1)$$

$$yPz \rightarrow N(yP_iz) \geq 1 \wedge N(yR_iz) = 2 \quad (2)$$

$(1) \wedge (2) \wedge zRx \rightarrow (\exists i, j \in L)[[xP_iyP_iz \wedge (zP_jx \wedge xI_jy \wedge zI_jy)] \vee [(xP_iy \wedge yI_iz \wedge xI_iz) \wedge (yP_jz \wedge zI_jx \wedge yI_jx)]]$

$\rightarrow$ LSIRR-Q is violated.

Suppose $D \subseteq \mathcal{Q}$ violates LSIRR-Q. Then there is some triple $A = \{x, y, z\}$ over which LSIRR-Q is violated. Violation of LSIRR-Q over the triple A implies that: $(\exists$ distinct $a, b, c \in \{x, y, z\})(\exists R^s, R^t \in D)[[(aP^sb \wedge bI^sc \wedge aI^sc) \wedge (bP^tc \wedge cI^ta \wedge bI^ta)] \vee [(aP^sb \wedge bI^sc \wedge aI^sc) \wedge bP^tcP^ta]]$. Consider any $(R_1, R_2) \in D^2$ such that $\#\{i \in L \mid R_i = R^s\} = \#\{i \in L \mid R_i = R^t\} = 1$. Then in each case MMD yields an R which violates quasi-transitivity.

5 Conditions for Acyclicity

Unlike transitivity and quasi-transitivity, condition of acyclicity is not defined over triples. Consequently, there is no reason to expect existence of conditions defined only over triples which can completely characterize all D, $D \subseteq \mathcal{B}$, $\mathcal{B} \subseteq \mathcal{C}$, which are such that all logically possible $(R_1, \ldots, R_n) \in D^n$ give rise to acyclic social $R, R = (R_1, \ldots, R_n)$, under the MMD. In fact, if $\mathcal{B} = \mathcal{Q}$, then the subsets $D \subseteq \mathcal{B}$ which are such that all logically possible $(R_1, \ldots, R_n) \in D^n$ give rise to acyclic social R under the MMD cannot be characterized by a condition defined only over triples as the following theorem shows.

Theorem 10 *Let f be the method of majority decision; and let $\#S = m \geq 4$ and $\#L = n \geq 2$. Let $\mathcal{D}_{\mathcal{Q}} = \{D \subseteq \mathcal{Q} \mid (\forall (R_1, \ldots, R_n) \in D^n)(R = f(R_1, \ldots, R_n)$ is acyclic$)\}$. Then, there does not exist any condition α defined only over triples such that $D, D \in 2^{\mathcal{Q}} - \{\emptyset\}$, belongs to $\mathcal{D}_{\mathcal{Q}}$ iff it satisfies condition α.*

Proof: Let condition α defined only over triples be such that $D, D \in 2^{\mathcal{Q}} - \{\emptyset\}$, belongs to $\mathcal{D}_{\mathcal{Q}}$ iff it satisfies condition α. Let $S = \{x, y, z, w, t_1, \ldots, t_{m-4}\}$. Consider $D = \{(xPy, yIz, xIz;\ x, y, zPwPt_1P \ldots Pt_{m-4}),\ (yPz, zIx, yIx;\ x, y, zPwPt_1P \ldots Pt_{m-4})\}$. It is immediate that the MMD yields acyclic R for every $(R_1, \ldots, R_n) \in D^n$; and consequently it follows that $D \in \mathcal{D}_{\mathcal{Q}}$. As condition α is defined only over triples, it follows that D must be satisfying α over every triple of alternatives. Therefore it follows that if $A \subseteq S$ is a triple and $(\exists$ distinct $a, b, c \in A)[D|\{a, b, c\} = \{(aPb \wedge bIc \wedge aIc), (aIb \wedge bPc \wedge aIc)\} \vee D|\{a, b, c\} = \{aPbPc, aIbPc\}]$ then D would satisfy α over A.

Now consider the following $(R_1, \ldots, R_n)$.
$(xP_1y, yI_1z, zP_1w, wI_1x, xI_1z, yI_1w;\ x, y, z, wP_1t_1P_1 \ldots P_1t_{m-4})$
$(\forall i \in L - \{1\})(xI_iy, yP_iz, zI_iw, wP_ix, xI_iz, yI_iw;\ x, y, z, wP_it_1P_i \ldots P_it_{m-4})$.
The R yielded by MMD for the above configuration is: $(xPy, yPz, zPw, wPx, xIz, yIw;$ $x, y, z, wPt_1P \ldots Pt_{m-4})$, which violates acyclicity. Now, for every triple of alternatives $A \subseteq S$ we have $(\exists$ distinct $a, b, c \in A)[\{R_i|A \mid i \in L\} = \{(aPb, bIc, aIc), (bPc, cIa, bIa)\} \vee \{R_i|A \mid i \in L\} = \{aPbPc, aIbPc\} \vee \{R_i|A \mid i \in L\} = \{aPbPc\} \vee \{R_i|A \mid i \in L\} = \{aIbPc\}$. Therefore, it follows that either $\{(aPb \wedge bIc \wedge aIc), (aIb \wedge bPc \wedge aIc)\}$ or $\{aPbPc, aIbPc\}$ must be violating α, contradicting the earlier conclusion that both of these sets satisfy α. This contradiction establishes the theorem.
From the above theorem the following corollary follows immediately.

Corollary 1 *Let f be the method of majority decision; and let $\#S = m \geq 4$ and $\#L = n \geq 2$. Let $\mathcal{D}_{\mathcal{A}} = \{D \subseteq \mathcal{A} \mid (\forall (R_1, \ldots, R_n) \in D^n)(R = f(R_1, \ldots, R_n)$ is acyclic$)\}$. Then, there does not exist any condition α defined only over triples such that $D, D \in 2^{\mathcal{A}} - \{\emptyset\}$, belongs to $\mathcal{D}_{\mathcal{A}}$ iff it satisfies condition α.*

If the number of individuals is greater than or equal to 11 then the sets $D \subseteq \mathcal{T}$ which are such that all logically possible $(R_1, \ldots, R_n) \in D^n$ give rise to acyclic social R under the MMD, however, can be characterized by a condition defined only over triples. The following theorem can easily be proved.

Theorem 11 *Let $\#S \geq 3$ and $\#L = n \geq 11$. Let $D \subseteq \mathcal{T}$. Then the method of majority decision f yields acyclic social $R, R = f(R_1, \ldots, R_n)$, for every $(R_1, \ldots, R_n) \in D^n$ iff D satisfies the condition of Latin Square partial agreement.*[4]

[4]See Sen and Pattanaik (1969) and Kelly (1974).

As far as the remaining cases are concerned, there is no uniformity among them. In some cases it can be shown that no condition defined only over triples can be a characterizing condition while in some other cases it is possible to formulate a characterizing condition defined only over the triples. For instance, it can be shown that if the number of individuals is 4 then there does not exist any condition defined only over triples which can characterize the sets of orderings which invariably give rise to acyclic social R. On the other hand the validity of Theorem 11 can be shown for $n = 9$ as well.[5]

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[5]See Kelly (1974).

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