The Method of Majority Decision: The Necessary and Sufficient Conditions for Transitivity and Quasi-Transitivity

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Abstract

This paper establishes the necessary and sufficient conditions for transitivity and quasi-transitivity under the method of majority decision. The terms 'necessary' and 'sufficient' here are used in the sense of logic, and not in the sense of usage common in the restricted domain literature. These necessary and sufficient conditions enable derivation of all existing theorems pertaining to transitivity and quasi-transitivity under the method of majority decision very simply, almost trivially.

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This paper establishes the necessary and sufficient conditions for transitivity and quasi-transitivity under the method of majority decision. The terms 'necessary' and 'sufficient' here are used in the sense of logic, and not in the sense of usage common in the restricted domain literature. To derive necessary and sufficient conditions, the notion of the reduced form of a profile of orderings is defined. The reduced form consists of occurrences of at most three linear orderings. Under the method of majority decision, social preferences generated by a profile are the same as those generated by the reduced form.

It is shown in the paper that under the method of majority decision a profile of orderings violates transitivity if and only if (a) Its reduced form has occurrences of only two linear orderings belonging to the same Latin Square and these two orderings have equal number of occurrences; or (b) Its reduced form has occurrences of three linear orderings belonging to the same Latin Square and the number of occurrences of none of them exceeds half the total number of occurrences. And, under the method of majority decision a profile of orderings violates quasi- transitivity if and only if (b) holds.

These necessary and sufficient conditions enable derivation of all existing theorems pertaining to transitivity and quasi-transitivity under the method of majority decision very simply, almost trivially.

*The author wishes to thank Kaushal Kishore for proofreading the paper and pointing out incompleteness of the argument in one place and several typographical errors.

The paper is divided into four sections. Section one contains the definitions and assumptions used in the paper; Section two defines and illustrates the notion of the reduced form; Section 3 contains the statement and proof of the theorem establishing a necessary and sufficient condition for transitivity under the method of majority decision; and Section 4 contains the statement and proof of the theorem establishing a necessary and sufficient condition for quasi-transitivity under the method of majority decision. The Appendix contains three sections: Section 5 contains definitions of two versions of Latin Squares and related concepts; Section 6 contains definitions of almost all restrictions on preferences that have figured in the literature in the context of the method of majority decision; and the final section contains alternative proofs of the standard theorems on the transitivity and quasi-transitivity under the method of majority decision using the necessary and sufficient conditions for transitivity and quasi-transitivity established here.

1 Definitions and Assumptions

Let $S = \{x, y, z\}$ be a set of three[1] distinct alternatives and let $N = \{1, 2, \ldots, n\}$ be the set of individuals, $n \geq 2, n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of positive integers. Each individual $i \in N$ is assumed to have an ordering R_i on S. R_i will be interpreted as 'at least as good as' for individual i. Asymmetric and symmetric parts of R_i will be denoted by P_i and I_i respectively. Therefore, P_i and I_i will have the interpretation as 'better than' and 'indifferent to' respectively from the perspective of individual i. Similarly, if R is a binary relation on S then P and I will denote the asymmetric and symmetric parts respectively of R. Let $\mathcal{T}$ denote the set of 13 logically possible orderings of S; and $\mathcal{C}$ the set of 27 logically possible reflexive and connected binary relations on S.[2]

13 logically possible orderings of S are:
(i) $xPyPz$ (ii) $yPzPx$ (iii) $zPxPy$ (iv) $xPzPy$ (v) $zPyPx$ (vi) $yPxPz$ (vii) $xPyIz$ (viii) $yPzIx$ (ix) $zPxIy$ (x) $xIyPz$ (xi) $yIzPx$ (xii) $zIxPy$ (xiii) $xIyIz$.

The following nomenclature will be used:

[1]The assumption that S has three alternatives is not a restrictive one as both transitivity and quasi-transitivity are conditions on triples. When S has more than three alternatives then the necessary and sufficient conditions derived in this paper must hold for all triple of alternatives for transitivity or quasi-transitivity, as the case may be, to hold over the entire set. The assumption of S having three alternatives has been made to avoid cluttering up of the notation.

[2]A binary relation R on S (i) reflexive iff $(\forall x \in S)(xRx)$; (ii) connected iff $(\forall x, y \in S)(x \neq y \rightarrow xRy \vee yRx)$; (iii) anti-symmetric iff $(\forall x, y \in S)(xRy \wedge yRx \rightarrow x = y)$; (iv) transitive iff $(\forall x, y, z \in S)(xRy \wedge yRz \rightarrow xRz)$; (v) quasi-transitive iff $(\forall x, y, z \in S)(xPy \wedge yPz \rightarrow xPz)$; (vi) an ordering iff it is reflexive, connected and transitive, and (vii) a linear ordering iff it is reflexive, connected, anti-symmetric, and transitive.

(a) Orderings (i)-(vi) will be called linear orderings.
(b) Orderings (i)-(xii) will be called concerned orderings.
(c) Orderings (vii)-(xii) will be called concerned non-linear orderings.
(d) Ordering (xiii) will be called unconcerned ordering.
(e) Orderings (i)-(iii) will be called linear orderings of Group I; and Orderings (iv)-(vi) will be called linear orderings of Group II.
(f) Orderings (i) and (v) are opposites of each other; orderings (ii) and (iv) are opposites of each other; and orderings (iii) and (vi) are opposites of each other.

We will also use abbreviations on the pattern: xyz for $xPyPz$; $x(yz)$ for $xPyIz$; $(xy)z$ for $xIyPz$; and (xyz) for $xIyIz$.

The Method of Majority Decision (MMD): MMD $f : \mathcal{T}^n \mapsto \mathcal{C}$ is defined by: $(\forall(R_1, \ldots, R_n) \in \mathcal{T}^n)(\forall x, y \in S)[xRy \leftrightarrow n(xP_iy) \geq n(yP_ix)]$, where $n()$ denotes the number of individuals having the preferences specified in the parentheses and R denotes the social binary relation determined by the MMD. P and I will be interpreted as 'socially better' and 'socially indifferent to' respectively.

From the definition of MMD it follows that $(\forall(R_1, \ldots, R_n) \in \mathcal{T}^n)(\forall x, y \in S)[[xPy \leftrightarrow n(xP_iy) > n(yP_ix)] \wedge [xIy \leftrightarrow n(xP_iy) = n(yP_ix)]]$. Thus under the method of majority decision, an alternative x is socially preferred to another alternative y iff the number of people who prefer x to y is greater than the number of people who prefer y to x; and x is socially indifferent to y iff the number of people who prefer x to y is equal to the number of people who prefer y to x.

2 The Reduced Form of a Profile

Let $(R_1, \ldots, R_n)$ be a profile. The reduced form of a profile is constructed sequentially and as follows:
(i) If a linear ordering occurs k times in the profile, its occurrences are doubled to $2k$ occurrences.
(ii) All occurrences of the unconcerned ordering xI_iyI_iz are deleted.
(iii) Every non-linear concerned ordering is replaced by two linear orderings such that both the linear orderings agree with the non-linear concerned ordering on the strict preferences occurring in it; and the linear orderings have opposite strict preferences over the pair in which indifference occurs in the non-linear concerned ordering.

If k is the number of unconcerned orderings in the profile then after the steps (i)-(iii) have been performed there will be $2(n-k)$ orderings; and all of them will be linear orderings.

(iv) For every pair of opposite linear orderings R^1 and R^2 with n^1 and n^2 occurrences respectively, $n^1, n^2 \in \mathbb{N}$, if $n^1 = n^2$ then delete all occurrences of R^1 and R^2; if $n^1 > n^2$, then delete all occurrences of R^2 and reduce the occurrences of R^1 by n^2; and if $n^1 < n^2$, then delete all occurrences of R^1 and reduce the occurrences of R^2 by n^1.

Thus after step (iv) occurrences of at most three linear orderings will be left.

It is clear from the construction of the reduced form that the social preferences generated by the MMD for $(R_1, \ldots, R_n)$ and for its reduced form would be identical.

The following example illustrates the procedure for constructing the reduced form.

Example 1 *Let* $S = \{x, y, z\}; N = \{1, 2, \ldots, 10\}$. *Consider the profile:* $(xP_1yP_1z, xP_2yI_2z, yP_3zI_3x, yP_4zI_4x, zP_5xI_5y, xI_6yP_6z, xI_7yI_7z, zP_8yP_8x, xP_9zP_9y, yP_{10}xP_{10}z)$. *By doubling occurrences of linear orderings* xyz, zyx, xzy, yxz, *deleting occurrence of the unconcerned ordering* (xyz), *and replacing non-linear concerned orderings* $x(yz), y(zx), y(zx), z(xy), (xy)z$ *by* $(xyz \wedge xzy), (yzx \wedge yxz), (yzx \wedge yxz), (zxy \wedge zyx), (xyz \wedge yxz)$ *respectively, we obtain:*

ordering	*occurrences*	*ordering*	*occurrences*
xyz	4	zyx	3
yzx	2	xzy	3
zxy	1	yxz	5

After applying step (iv) we will be left with:

ordering	*occurrences*
xyz	1
xzy	1
yxz	4

Thus the reduced form consists of 1 occurrence of xyz, *one occurrence of* xzy, *and 4 occurrences of* yxz.

For the given profile we have: $n(xP_iy) = 3, n(yP_ix) = 4, n(yP_iz) = 5, n(zP_iy) = 3, n(xP_iz) = 5, n(zP_ix) = 2$. *Therefore, under the MMD the profile yields* $yPx \wedge yPz \wedge xPz$; *and so does the reduced profile.* $\Diamond$

3 Necessary and Sufficient Condition for Transitivity

Theorem 1 *Let $S = \{x, y, z\}; N = \{1, 2, \ldots, n\}, n \geq 2, n \in \mathbb{N}$. Let f be the method of majority decision. Then a profile of individual orderings $(R_1, \ldots, R_n)$ does not yield transitive social $R, R = f(R_1, \ldots, R_n)$, under the MMD iff (a) The reduced form has occurrences of only two linear orderings belonging to the same group and these two orderings have equal number of occurrences; or (b) The reduced form has occurrences of three linear orderings belonging to the same group and the number of occurrences of none of them exceeds half the total number of occurrences.*

Proof: Sufficiency

(a) Let the reduced form of the profile $(R_1, \ldots, R_n)$ consist of occurrences of only two linear orderings of the same group with the two orderings having equal number of occurrences. Without any loss of generality assume that these orderings are xyz and yzx and that each occurs k times. Then the social preferences corresponding to the reduced form, and consequently corresponding to the profile $(R_1, \ldots, R_n)$, under the MMD would be $(xIy \wedge yPz \wedge xIz)$ violating transitivity.
(b) Let the reduced form of the profile $(R_1, \ldots, R_n)$ consist of occurrences of three linear orderings belonging to the same group and let the number of occurrences of none of them exceed half the total number of occurrences. Without any loss of generality assume that these orderings are xyz, yzx, zxy and they occur n_1, n_2, n_3 times respectively; and that $n_1, n_2, n_3 \leq \frac{1}{2}(n_1 + n_2 + n_3)$. If each of n_1, n_2, n_3 is less than $\frac{1}{2}(n_1 + n_2 + n_3)$, then we obtain $(xPy \wedge yPz \wedge zPx)$. If one of them, say xyz, has $\frac{1}{2}(n_1 + n_2 + n_3)$ occurrences then we obtain $(xPy \wedge yPz \wedge xIz)$. Transitivity is violated in either case.

Necessity

Let the reduced form of the profile $(R_1, \ldots, R_n)$ be such that neither of (a) and (b) mentioned in the statement of the Theorem holds. Then it must be the case that:
(i) The reduced form has no orderings; or
(ii) The reduced form has occurrences of just one linear ordering; or
(iii) The reduced form has occurrences of only two linear orderings, one belonging to Group I, and one belonging to Group II; or
(iv) The reduced form has occurrences of just two linear orderings belonging to the same group and the two orderings have unequal number of occurrences; or
(v) The reduced form has occurrences of three linear orderings, two orderings belonging to one group, and one ordering belonging to the other group; or
(vi) The reduced form has occurrences of three linear orderings belonging to the same

group, and one of them has more than half of the total occurrences of all three orderings.

(i) If there are no orderings in the reduced form then the social R under the MMD is $(xIy \wedge yIz \wedge xIz)$.
(ii) If there are occurrences of only one linear ordering in the reduced form then the social R under the MMD is identical with the linear ordering.
(iii) Let the reduced form consist of occurrences of two linear orderings, one belonging to Group I, and one belonging to Group II. Without any loss of generality assume that the linear ordering belonging to Group I is xyz. In view of the Step 4 of the construction of the reduced form, the linear ordering belonging to Group II cannot be zyx; so it must be either xzy or yxz. We consider both the cases. If the orderings are $(xyz \wedge xzy)$ then under the MMD we must have $(xPy \wedge xPz)$ implying transitive R regardless of the social preferences between y and z. If the orderings are $(xyz \wedge yxz)$ then under the MMD we must have $(yPz \wedge xPz)$ implying transitive R regardless of the social preferences between x and y.
(iv) Let the reduced form have occurrences of just two linear orderings belonging to the same group and let the two orderings have unequal number of occurrences. Then under the MMD the social R will be identical to the linear ordering having the larger number of occurrences.
(v) The reduced form has occurrences of three linear orderings, two orderings belonging to one group, and one ordering belonging to the other group. Without any loss of generality assume two orderings belong to Group I and one ordering to Group II. Without any loss of generality assume that the two orderings belonging to Group I are xyz and yzx. In view of Step 4 of the construction of the reduced form, the linear ordering belonging to Group II then must be yxz. Let the occurrences of these orderings xyz, yzx, yxz be n_1, n_2, n_3 respectively. As in all three orderings we have y preferred to z, it is impossible to have $(xRz \wedge zRy \wedge yRx)$. As $(zRx \wedge xRy)$ would imply that n_3 is zero, it follows that $(xRy \wedge yRz \wedge zRx)$ is also impossible. Thus social R is necessarily transitive.
(vi) The reduced form has occurrences of three linear orderings belonging to the same group, and one of them has more than half of the total occurrences of all three orderings. Under the MMD then the social R would coincide with the linear ordering having more than half of the total occurrences of all three orderings.

Thus in all 6 cases transitivity holds. $\square$

4 Necessary and Sufficient Condition for Quasi-Transitivity

Theorem 2 *Let $S = \{x, y, z\}; N = \{1, 2, \ldots, n\}, n \geq 2, n \in \mathbb{N}$. Let f be the method of majority decision. Then a profile of individual orderings $(R_1, \ldots, R_n)$ does not yield quasi-transitive social $R, R = f(R_1, \ldots, R_n)$, under the MMD iff (b) of Theorem 1 holds, i.e., the reduced form has occurrences of three linear orderings belonging to the same group and the number of occurrences of none of them exceeds half the total number of occurrences.*

Proof: Sufficiency

Let the reduced form of the profile $(R_1, \ldots, R_n)$ consist of occurrences of three linear orderings belonging to the same group and let the number of occurrences of none of them exceed half the total number of occurrences. Without any loss of generality assume that these orderings are xyz, yzx, zxy and they occur n_1, n_2, n_3 times respectively; and that $n_1, n_2, n_3 \leq \frac{1}{2}(n_1 + n_2 + n_3)$. If each of n_1, n_2, n_3 is less than $\frac{1}{2}(n_1 + n_2 + n_3)$, then we obtain $(xPy \wedge yPz \wedge zPx)$. If one of them, say xyz, has $\frac{1}{2}(n_1 + n_2 + n_3)$ occurrences then we obtain $(xPy \wedge yPz \wedge xIz)$. Quasi-transitivity is violated in either case.

Necessity

Let the reduced form of the profile $(R_1, \ldots, R_n)$ be such that it does not consist of occurrences of three linear orderings belonging to the same group with the number of occurrences of none of them exceeding half the total number of occurrences Then it must be the case that:
(i) The reduced form has no orderings; or
(ii) The reduced form has occurrences of just one linear ordering; or
(iii) The reduced form has occurrences of only two linear orderings, one belonging to Group I, and one belonging to Group II; or
(iv) The reduced form has occurrences of just two linear orderings belonging to the same group and the two orderings have unequal number of occurrences; or
(v) The reduced form has occurrences of three linear orderings, two orderings belonging to one group, and one ordering belonging to the other group; or
(vi) The reduced form has occurrences of three linear orderings belonging to the same group, and one of them has more than half of the total occurrences of all three orderings.
(vii) The reduced form has occurrences of just two linear orderings belonging to the same group and the two orderings have equal number of occurrences.

In Theorem 1 it has been shown that if any of (i)-(vi) holds then the social preferences yielded by the MMD are transitive, and therefore quasi-transitive. Therefore it suffices to consider only (vii).

Let the reduced form consist of occurrences of just two linear orderings belonging to the same group and let the two orderings have equal number of occurrences. Without any loss of generality, assume that these orderings are xyz and yzx. Then the social preferences under the MMD are $(xIy \wedge yPz \wedge xIz)$, satisfying quasi-transitivity.

Thus in all 7 cases quasi-transitivity holds. $\square$

Appendix

5 Latin Squares

We define in S, according to ordering R, x to be best iff $(xRy \wedge xRz)$; to be medium iff $(yRxRz \vee zRxRy)$; to be worst iff $(yRx \wedge zRx)$.

Weak Latin Square (WLS): Let R^s, R^t, R^u be orderings on S. The set $\{R^s, R^t, R^u\}$ forms a weak Latin Square over S iff $(\exists$ distinct $a, b, c \in S)[aR^sbR^sc \wedge bR^tcR^ta \wedge cR^uaR^ub]$. The above weak Latin Square will be denoted by $WLS(abca)$.

Remark 1 *R^s, R^t, R^u in the definition of weak Latin Square need not be distinct. $\{xIyIz\}$ forms a weak Latin Square over $\{x, y, z\}$.* $\Diamond$

Latin Square (LS): Let R^s, R^t, R^u be orderings on S. The set $\{R^s, R^t, R^u\}$ forms a Latin Square over S iff $[(R^s, R^t, R^u$ are concerned$) \wedge (\exists$ distinct $a, b, c \in S)[aR^sbR^sc \wedge bR^tcR^ta \wedge cR^uaR^ub]]$. The above Latin Square will be denoted by $LS(abca)$.

We define:
$T[WLS(abca)] = \{R \in \mathcal{T} \mid aRbRc \vee bRcRa \vee cRaRb\}$.
$T[LS(abca)] = \{R \in \mathcal{T} \mid\ R$ is concerned $\wedge (aRbRc \vee bRcRa \vee cRaRb)\}$.

Thus we have:
$T[WLS(xyzx)] = T[WLS(yzxy)] = T[WLS(zxyz)] = \{xPyPz,\ xPyIz,\ xIyPz,$
$yPzPx,\ yPzIx,\ yIzPx,\ zPxPy,\ zPxIy,\ zIxPy,\ xIyIz\}$
$T[WLS(xzyx)] = T[WLS(zyxz)] = T[WLS(yxzy)] = \{xPzPy,\ xPzIy,\ xIzPy,$
$zPyPx,\ zPyIx,\ zIyPx,\ yPxPz,\ yPxIz,\ yIxPz,\ xIyIz\}$
$T[LS(xyzx)] = T[LS(yzxy)] = T[LS(zxyz)] = T[WLS(xyzx)] - \{xIyIz\}$

$T[LS(xzyx)] = T[LS(zyxz)] = T[LS(yxzy)] = T[WLS(xzyx)] - \{xIyIz\}$.

6 Domain Restriction Conditions

Let L be a linear ordering of S. We define x to be between y and z, denoted by $B_L(y, x, z)$, iff $[(yLx \wedge xLz) \vee (zLx \wedge xLy)]$.

Single Peakedness (SP): $\mathcal{D} \subseteq \mathcal{T}$ satisfies SP iff ($\exists$ a linear ordering L of S)($\forall$ $a, b, c \in S$)($\forall R \in \mathcal{D}$)$[aRb \wedge B_L(a, b, c) \rightarrow bPc]$.

Remark 2 *From the definition of SP it is clear that a set of orderings of S satisfies the condition of single-peakedness iff there is an alternative such that it is not worst in any of the orderings of the set. Thus, except for a permutation of alternatives, the maximal set of orderings satisfying SP is given by:* $\{xyz, zxy, xzy, yxz, x(yz), (xy)z, (zx)y\}$. $\diamondsuit$

Single Cavedness (SC): $\mathcal{D} \subseteq \mathcal{T}$ satisfies SC iff ($\exists$ a linear ordering L of S)($\forall$ $a, b, c \in S$)($\forall R \in \mathcal{D}$)$[bRa \wedge B_L(a, b, c) \rightarrow cPb]$.

Remark 3 *A set of orderings of S satisfies the condition of single-cavedness iff there is an alternative such that it is not best in any of the orderings of the set. Thus, except for a permutation of alternatives, the maximal set of orderings satisfying SC is given by:* $\{yzx, zxy, zyx, yxz, y(zx), z(xy), (yz)x\}$. $\diamondsuit$

Separability into Two Groups (SG): $\mathcal{D} \subseteq \mathcal{T}$ satisfies SG iff $(\exists S_1, S_2 \subset S)[[S_1 \neq \emptyset \wedge S_2 \neq \emptyset \wedge S_1 \cap S_2 = \emptyset \wedge S_1 \cup S_2 = S] \wedge (\forall R \in \mathcal{D})[(\forall a \in S_1)(\forall b \in S_2)(aPb)] \vee (\forall a \in S_1)(\forall b \in S_2)(bPa)]$.

Remark 4 *A set of orderings of S satisfies the condition of separability into two groups iff there is an alternative such that it is not medium in any of the orderings of the set. Thus, except for a permutation of alternatives, the maximal set of orderings satisfying SG is given by:* $\{xyz, yzx, xzy, zyx, x(yz), (yz)x\}$. $\diamondsuit$

First Version of Value-Restriction (VR(1)): $\mathcal{D} \subseteq \mathcal{T}$ satisfies VR(1) iff ($\exists$ distinct $a, b, c \in S$)$[(\forall R \in \mathcal{D})[bPa \vee cPa] \vee (\forall R \in \mathcal{D})[(aPb \wedge aPc) \vee (bPa \wedge cPa)] \vee (\forall R \in \mathcal{D})[aPb \vee aPc]]$. In other words, a set of orderings of S satisfies VR(1) iff it satisfies SP or SC or SG.

Remark 5 *As VR(1) is the union of SP, SC, SG, the maximal sets of orderings satisfying VR(1) are simply the maximal sets of orderings of SP, SC, SG.* $\diamondsuit$

Remark 6 *A set of orderings of S violates value-restriction (1) iff it violates all three conditions SP, SC, SG. It can easily be checked that VR(1) is violated iff there is a weak Latin Square.* $\diamondsuit$

Second Version of Value-Restriction (VR(2)): $\mathcal{D} \subseteq \mathcal{T}$ satisfies VR(2) iff $(\exists$ distinct $a, b, c \in S)[(\forall$ concerned $R \in \mathcal{D})[bPa \vee cPa] \vee (\forall$ concerned $R \in \mathcal{D})[(aPb \wedge aPc) \vee (bPa \wedge cPa)] \vee (\forall$ concerned $R \in \mathcal{D})[aPb \vee aPc]]$.

Remark 7 *The only difference between VR(1) and VR(2) is that (xyz) is excluded by VR(1) but not by VR(2). The maximal sets of VR(2) are obtained by including (xyz) in the maximal sets of VR(1).* ◊

Remark 8 *A set of orderings of S violates value-restriction (2) iff there is a Latin Square.* ◊

Dichotomous Preferences (DP): $\mathcal{D} \subseteq \mathcal{T}$ satisfies DP iff $\sim (\exists$ distinct $a, b, c \in S)(\exists R \in \mathcal{D})[aPbPc]$.

Remark 9 *The maximal set of orderings satisfying DP is given by:*
$\{x(yz), y(zx), z(xy), (xy)z, (yz)x, (zx)y, (xyz)\}$. ◊

Echoic Preferences (EP): $\mathcal{D} \subseteq \mathcal{T}$ satisfies EP iff $(\forall$ distinct $a, b, c \in S)[aPbPc \in \mathcal{D} \rightarrow (\forall R \in \mathcal{D})(aRc)]$.

Remark 10 *Leaving out the maximal set corresponding to DP, there are three maximal sets of orderings satisfying EP, except for a permutation of alternatives. These are:*
$\{xyz, x(yz), y(zx), (xy)z, (zx)y, (xyz)\}$
$\{xyz, xzy, x(yz), (xy)z, (zx)y, (xyz)\}$
$\{xyz, yxz, x(yz), y(zx), (xy)z, (xyz)\}$. ◊

Antagonistic Preferences (AP): $\mathcal{D} \subseteq \mathcal{T}$ satisfies AP iff $(\forall$ distinct $a, b, c \in S)[aPbPc \in \mathcal{D} \rightarrow (\forall R \in \mathcal{D})(aPbPc \vee cPbPa \vee aIc)]$.

Remark 11 *Leaving out the maximal set corresponding to DP, there is only one maximal set of orderings satisfying AP, except for a permutation of alternatives. The set is:*
$\{xPyPz, zPyPx, zIxPy, yPxIz, xIyIz\}$. ◊

Extremal Restriction (ER): $\mathcal{D} \subseteq \mathcal{T}$ satisfies ER iff $(\forall$ distinct $a, b, c \in S)[(\exists R \in \mathcal{D})(aPbPc) \rightarrow (\forall R \in \mathcal{D} \cap T[LS(abca)])(aRc)]$.

Remark 12 *Extremal restriction is usually defined as follows: $\mathcal{D} \subseteq \mathcal{T}$ satisfies ER iff $(\forall$ distinct $a, b, c \in S)[(\exists R \in \mathcal{D})(aPbPc) \rightarrow (\forall R \in \mathcal{D})(\sim cPa \vee cPbPa)]$. It is clear that the two definitions are equivalent to each other.* ◊

Remark 13 *As ER is the union of DP, EP, AP, the maximal sets of orderings satisfying ER are those of DP, EP, AP. Thus, except for a permutation of alternatives, there are 5 maximal sets satisfying ER.* ◊

Taboo Preferences (TP): $\mathcal{D} \subseteq \mathcal{T}$ satisfies TP iff $xIyIz \notin \mathcal{D} \wedge (\exists$ distinct $a, b \in S)(\forall R \in \mathcal{D})(aRb)$.

Remark 14 *Except for a permutation of alternatives, the maximal set of orderings satisfying TP is:* $\{(xy)z, yzx, y(zx), (yz)x, z(xy), yxz, zyx\}$. $\Diamond$

Weak Latin Square Partial Agreement (WLSPA): $\mathcal{D} \subseteq \mathcal{T}$ satisfies WLSPA iff $(\forall$ distinct $a, b, c \in S)[(\exists R^s, R^t, R^u \in \mathcal{D})(aP^sbP^sc \wedge bR^tcR^ta \wedge cR^uaR^ub) \rightarrow (\forall R \in \mathcal{D} \cap T[LS(abca)])(aRc)]$.

Remark 15 *As WLSPA is the union of VR(1), ER, TP, the maximal sets of orderings satisfying WLSPA are those of VR(1), ER, TP. Thus, except for a permutation of alternatives, there are 9 maximal sets satisfying WLSPA.* $\Diamond$

Limited Agreement (LA): $\mathcal{D} \subseteq \mathcal{T}$ satisfies LA iff $(\exists$ distinct $a, b \in S)(\forall R \in \mathcal{D})(aRb)$.

Remark 16 *Except for a permutation of alternatives, the maximal set of orderings satisfying LA is:* $\{(xy)z, yzx, y(zx), (yz)x, z(xy), yxz, zyx, (xyz)\}$ *This set is the same as that of TP except for the addition of* (xyz)*.* $\Diamond$

Latin Square Partial Agreement (LSPA): $\mathcal{D} \subseteq \mathcal{T}$ satisfies LSPA iff $(\forall$ distinct $a, b, c \in S)[(\exists R^s, R^t, R^u \in \mathcal{D})(R^s, R^t, R^u$ are concerned $\wedge aP^sbP^sc \wedge bR^tcR^ta \wedge cR^uaR^ub) \rightarrow (\forall R \in \mathcal{D} \cap T[LS(abca)])(aRc)]$.

Remark 17 *As LSPA is the union of VR(2), LA, DP, AP, the maximal sets of orderings satisfying LSPA are those of VR(2), LA, DP, AP. Thus, except for a permutation of alternatives, there are 6 maximal sets satisfying LSPA. LSPA is also equivalent to the union of VR(2), LA, ER.* $\Diamond$

Weak Extremal Restriction (WER): $\mathcal{D} \subseteq \mathcal{T}$ satisfies WER iff $\sim (\exists$ distinct $a, b, c \in S)(\exists R^s, R^t, R^u \in \mathcal{D})(aP^sbP^sc \wedge bR^tcP^ta \wedge cP^uaR^ub)$.

Latin Square Linear Ordering Restriction (LSLOR): $\mathcal{D} \subseteq \mathcal{T}$ satisfies LSLOR iff $\sim (\exists$ distinct $a, b, c \in S)(\exists R^s, R^t, R^u \in \mathcal{D})(R^s, R^t, R^u$ are concerned over $A \wedge aP^sbP^sc \wedge bP^tcP^ta \wedge cR^uaR^ub)$.

Remark 18 *From the definitions it is clear that ER implies WLSPA; WLSPA implies LSPA; LSPA implies WER; and WER implies LSLOR.* $\Diamond$

7 Alternative Proofs of Standard Theorems Using the Characterizing Conditions of Theorems 1 and 2

Proposition 1 *If a profile $(R_1, \ldots, R_n)$ satisfies the condition of extremal restriction then social R corresponding to it generated by the method of majority decision is transitive.*

Proof: Let profile $(R_1, \ldots, R_n)$ satisfy ER. Then the orderings in the profile must be a subset of one of the five maximal sets listed in Remarks 9-11 (see Remark 13). We consider each of these five cases.

(I) Let the orderings in the profile be a subset of $\{x(yz), y(zx), z(xy), (xy)z, (yz)x, (zx)y, (xyz)\}$. Let the number of individuals holding the orderings 1. $x(yz)$ 2. $y(zx)$ 3. $z(xy)$ 4. $(xy)z$ 5. $(yz)x$ 6. $(zx)y$ 7. (xyz) be $n_1, n_2, n_3, n_4, n_5, n_6, n_7$ respectively. Replace each occurrence of $x(yz)$ by $(xyz \wedge xzy)$, of $y(zx)$ by $(yzx \wedge yxz)$, of $z(xy)$ by $(zxy \wedge zyx)$, of $(xy)z$ by $(xyz \wedge yxz)$, of $(yz)x$ by $(yzx \wedge zyx)$, of $(zx)y$ by $(zxy \wedge xzy)$; and delete all occurrences of (xyz). Then we obtain:

ordering	*occurrences*	*ordering*	*occurrences*	*difference*
xyz	$n_1 + n_4$	zyx	$n_3 + n_5$	$d_1 = n_1 + n_4 - n_3 - n_5$
yzx	$n_2 + n_5$	xzy	$n_1 + n_6$	$d_2 = n_2 + n_5 - n_1 - n_6$
zxy	$n_3 + n_6$	yxz	$n_2 + n_4$	$d_3 = n_3 + n_6 - n_2 - n_4$

As (i) it is not possible for all three d_1, d_2, d_3 to be positive; (ii) it is not possible for all three d_1, d_2, d_3 to be negative; (iii) if any two of d_1, d_2, d_3 are positive then the third one is negative; and (iv) if any two of d_1, d_2, d_3 are negative then the third one is positive, it follows that neither (a) nor (b) of Theorem 1 can hold.

(II) Let the orderings in the profile be a subset of $\{xyz, x(yz), y(zx), (xy)z, (zx)y, (xyz)\}$. Let the number of individuals holding the orderings 1. xyz 2. $x(yz)$ 3. $y(zx)$ 4. $(xy)z$ 5. $(zx)y$ 6. (xyz) be $n_1, n_2, n_3, n_4, n_5, n_6$ respectively. Double the occurrences of xyz, replace each occurrence of $x(yz)$ by $(xyz \wedge xzy)$, of $y(zx)$ by $(yzx \wedge yxz)$, of $(xy)z$ by $(xyz \wedge yxz)$, of $(zx)y$ by $(zxy \wedge xzy)$; and delete all occurrences of (xyz). Then we obtain:

ordering	*occurrences*	*ordering*	*occurrences*	*difference*
xyz	$2n_1 + n_2 + n_4$	zyx	0	$d_1 = 2n_1 + n_2 + n_4$
yzx	n_3	xzy	$n_2 + n_5$	$d_2 = n_3 - n_2 - n_5$
zxy	n_5	yxz	$n_3 + n_4$	$d_3 = n_5 - n_3 - n_4$

(i) It is not possible for the reduced form to consist of occurrences of three linear orderings of Group II as occurrences of zyx are 0. (ii) It is not possible for the reduced form to consist of occurrences of two or three linear orderings of Group I as it is not possible to have $d_2 \geq 0 \wedge d_3 \geq 0 \wedge (d_2 > 0 \vee d_3 > 0)$. (iii) It is not possible for the reduced form to consist of occurrences of two linear orderings of Group II. (i)-(iii) imply that neither (a) nor (b) of Theorem 1 can hold.

(III) Let the orderings in the profile be a subset of $\{xyz, xzy, x(yz), (xy)z, (zx)y, (xyz)\}$. Let the number of individuals holding the orderings 1. xyz 2. xzy 3. $x(yz)$ 4. $(xy)z$ 5. $(zx)y$ 6. (xyz) be $n_1, n_2, n_3, n_4, n_5, n_6$ respectively. Double the occurrences of xyz and xzy, replace each occurrence of $x(yz)$ by $(xyz \wedge xzy)$, of $(xy)z$ by $(xyz \wedge yxz)$, of $(zx)y$ by $(zxy \wedge xzy)$; and delete all occurrences of (xyz). Then we obtain:

ordering	*occurrences*	*ordering*	*occurrences*	*difference*
xyz	$2n_1 + n_3 + n_4$	zyx	0	$d_1 = 2n_1 + n_3 + n_4$
yzx	0	xzy	$2n_2 + n_3 + n_5$	$d_2 = -2n_2 - n_3 - n_5$
zxy	n_5	yxz	n_4	$d_3 = n_5 - n_4$

(i) It is not possible for the reduced form to consist of occurrences of three linear orderings of Group I as occurrences of yzx are 0. (ii) It is not possible for the reduced form to consist of occurrences of three linear orderings of Group II as occurrences of zyx are 0. (iii) It is not possible for the reduced form to consist of occurrences of two linear orderings of one of the two groups. (i)-(iii) imply that neither (a) nor (b) of Theorem 1 can hold.

(IV) Let the orderings in the profile be a subset of $\{xyz, yxz, x(yz), y(zx), (xy)z, (xyz)\}$. Let the number of individuals holding the orderings 1. xyz 2. yxz 3. $x(yz)$ 4. $y(zx)$ 5. $(xy)z$ 6. (xyz) be $n_1, n_2, n_3, n_4, n_5, n_6$ respectively. Double the occurrences of xyz and yxz, replace each occurrence of $x(yz)$ by $(xyz \wedge xzy)$, of $y(zx)$ by $(yzx \wedge yxz)$, of $(xy)z$ by $(xyz \wedge yxz)$; and delete all occurrences of (xyz). Then we obtain:

ordering	*occurrences*	*ordering*	*occurrences*	*difference*
xyz	$2n_1 + n_3 + n_5$	zyx	0	$d_1 = 2n_1 + n_3 + n_5$
yzx	n_4	xzy	n_3	$d_2 = n_4 - n_3$
zxy	0	yxz	$2n_2 + n_4 + n_5$	$d_3 = -2n_2 - n_4 - n_5$

As occurrences of zxy and zyx are 0, it is not possible for the reduced form to consist of occurrences of three linear orderings of the same group. Also, it is not possible for the reduced form to consist of occurrences of two linear orderings of the same group. Thus neither (a) nor (b) of Theorem 1 can hold.

(V) Let the orderings in the profile be a subset of $\{xyz, zyx, y(zx), (zx)y, (xyz)\}$. Let the number of individuals holding the orderings 1. xyz 2. zyx 3. $y(zx)$ 4. $(zx)y$ 5. (xyz) be n_1, n_2, n_3, n_4, n_5 respectively. Double the occurrences of xyz and zyx, replace each occurrence of $y(zx)$ by $(yzx \wedge yxz)$, of $(zx)y$ by $(zxy \wedge xzy)$; and delete all occurrences of (xyz). Then we obtain:

ordering	*occurrences*	*ordering*	*occurrences*	*difference*
xyz	$2n_1$	zyx	$2n_2$	$d_1 = 2n_1 - 2n_2$
yzx	n_3	xzy	n_4	$d_2 = n_3 - n_4$
zxy	n_4	yxz	n_3	$d_3 = n_4 - n_3$

If $d_2 > 0$ then $d_3 < 0$; if $d_2 < 0$ then $d_3 > 0$; and if $d_2 = 0$ then $d_3 = 0$. Therefore, it is not possible for the reduced form to consist of occurrences of two or three linear orderings of the same group. Thus neither (a) nor (b) of Theorem 1 can hold.

This completes the proof of the proposition. □

Proposition 2 *Let $n = 2k, k \in \mathbb{N}$. Let $\mathcal{D} \subseteq \mathcal{T}$ be a profile of orderings of S violating ER. Then there exists a profile $(R_1, \ldots, R_n) \in \mathcal{D}^n$ which under the MMD generates intransitive social R.*

Proof: Let $\mathcal{D} \subseteq \mathcal{T}$ violate ER. From the definition of ER, it follows that $\mathcal{D}$ must contain $[xyz \wedge [yzx \vee (yz)x \vee zxy \vee z(xy)]]$, except for a permutation of alternatives. Consider profile $(R_1, \ldots, R_n)$ such that [k individuals have ordering $xyz \wedge$ [k individuals have ordering $yzx \vee k$ individuals have ordering $(yz)x \vee k$ individuals have ordering $zxy \vee k$ individuals have ordering $z(xy)$]. In each of the 4 cases the reduced form consists of occurrences of two linear orderings of the same group with equal number of occurrences. Thus (a) of Theorem 1 holds. □

Combining Propositions 1 and 2 we obtain:

Theorem 3 *Let $n = 2k, k \in \mathbb{N}$. Let $\mathcal{D} \subseteq \mathcal{T}$. Then every profile $(R_1, \ldots, R_n) \in \mathcal{D}^n$ yields transitive social R under the MMD iff $\mathcal{D}$ satisfies ER.*

Proposition 3 *(i) If a profile $(R_1, \ldots, R_n)$ satisfies VR(2) then social R corresponding to it generated by the MMD is quasi-transitive.*
(ii) If $n = 2k + 1, k \in \mathbb{N}$, then if a profile $(R_1, \ldots, R_n)$ satisfies VR(1) then social R corresponding to it generated by the MMD is transitive.

Proof: Let profile $(R_1, \ldots, R_n)$ satisfy VR(2). Then the orderings in the profile must be a subset of union of (xyz) and one of the three maximal sets listed in Remarks 2, 3, and 4 (see also Remarks 5 and 7). We consider each of these three cases.

(I) Let the orderings in the profile be a subset of $\{xyz, zxy, xzy, yxz, x(yz), (xy)z, (zx)y, (xyz)\}$. Let the number of individuals holding the orderings 1. xyz 2. zxy 3. xzy 4. yxz 5. $x(yz)$ 6. $(xy)z$ 7. $(zx)y$ 8. (xyz) be $n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8$ respectively. Double the occurrences of xyz, zxy, xzy, yxz; replace each occurrence of $x(yz)$ by $(xyz \wedge xzy)$, of $(xy)z$ by

$(xyz \wedge yxz)$, of $(zx)y$ by $(zxy \wedge xzy)$; and delete all occurrences of (xyz). Then we obtain:

ordering	*occurrences*	*ordering*	*occurrences*	*difference*
xyz	$2n_1 + n_5 + n_6$	zyx	0	$d_1 = 2n_1 + n_5 + n_6$
yzx	0	xzy	$2n_3 + n_5 + n_7$	$d_2 = -2n_3 - n_5 - n_7$
zxy	$2n_2 + n_7$	yxz	$2n_4 + n_6$	$d_3 = 2n_2 + n_7 - 2n_4 - n_6$

As occurrences of yzx and zyx are 0, (b) of Theorem 1 cannot hold.
If $n = 2k+1, k \in \mathbb{N}$ and $n_8 = 0$, then (a) of Theorem 1 cannot hold; as $[d_1 > 0 \wedge d_2 = 0 \wedge d_3 > 0 \wedge d_1 = d_3 \rightarrow n \text{ is even}] \wedge [d_1 = 0 \wedge d_2 < 0 \wedge d_3 < 0 \wedge d_2 = d_3 \rightarrow n \text{ is even}]$.

(II) Let the orderings in the profile be a subset of $\{yzx, zxy, zyx, yxz, y(zx), z(xy), (yz)x, (xyz)\}$. Let the number of individuals holding the orderings 1. yzx 2. zxy 3. zyx 4. yxz 5. $y(zx)$ 6. $z(xy)$ 7. $(yz)x$ 8. (xyz) be $n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8$ respectively. Double the occurrences of yzx, zxy, zyx, yxz; replace each occurrence of $y(zx)$ by $(yzx \wedge yxz)$, of $z(xy)$ by $(zxy \wedge zyx)$, of $(yz)x$ by $(yzx \wedge zyx)$; and delete all occurrences of (xyz). Then we obtain:

ordering	*occurrences*	*ordering*	*occurrences*	*difference*
xyz	0	zyx	$2n_3 + n_6 + n_7$	$d_1 = -2n_3 - n_6 - n_7$
yzx	$2n_1 + n_5 + n_7$	xzy	0	$d_2 = 2n_1 + n_5 + n_7$
zxy	$2n_2 + n_6$	yxz	$2n_4 + n_5$	$d_3 = 2n_2 + n_6 - 2n_4 - n_5$

As occurrences of xyz and xzy are 0, (b) of Theorem 1 cannot hold.
If $n = 2k+1, k \in \mathbb{N}$ and $n_8 = 0$, then (a) of Theorem 1 cannot hold; as $[d_1 = 0 \wedge d_2 > 0 \wedge d_3 > 0 \wedge d_2 = d_3 \rightarrow n \text{ is even}] \wedge [d_1 < 0 \wedge d_2 = 0 \wedge d_3 < 0 \wedge d_1 = d_3 \rightarrow n \text{ is even}]$.

(III) Let the orderings in the profile be a subset of $\{xyz, yzx, xzy, zyx, x(yz), (yz)x, (xyz)\}$. Let the number of individuals holding the orderings 1. xyz 2. yzx 3. xzy 4. zyx 5. $x(yz)$ 6. $(yz)x$ 7. (xyz) be $n_1, n_2, n_3, n_4, n_5, n_6, n_7$ respectively. Double the occurrences of xyz, yzx, xzy, zyx; replace each occurrence of $x(yz)$ by $(xyz \wedge xzy)$, of $(yz)x$ by $(yzx \wedge zyx)$; and delete all occurrences of (xyz). Then we obtain:

ordering	*occurrences*	*ordering*	*occurrences*	*difference*
xyz	$2n_1 + n_5$	zyx	$2n_4 + n_6$	$d_1 = 2n_1 + n_5 - 2n_4 - n_6$
yzx	$2n_2 + n_6$	xzy	$2n_3 + n_5$	$d_2 = 2n_2 + n_6 - 2n_3 - n_5$
zxy	0	yxz	0	0

As occurrences of zxy and yxz are 0, (b) of Theorem 1 cannot hold.
If $n = 2k+1, k \in \mathbb{N}$ and $n_7 = 0$, then (a) of Theorem 1 cannot hold; as $[d_1 > 0 \wedge d_2 > 0 \wedge d_1 = d_2 \rightarrow n \text{ is even}] \wedge [d_1 < 0 \wedge d_2 < 0 \wedge d_1 = d_2 \rightarrow n \text{ is even}]$.

Thus the proposition is established. □

Proposition 4 *(i) If a profile $(R_1, \ldots, R_n)$ satisfies limited agreement then social R cor-*

responding to it generated by the MMD is quasi-transitive.
(ii) If $n = 2k+1, k \in \mathbb{N}$, then if a profile $(R_1, \ldots, R_n)$ satisfies taboo preferences then social R corresponding to it generated by the MMD is transitive.

Proof: Let profile $(R_1, \ldots, R_n)$ satisfy LA. Then the orderings in the profile must be a subset of $\{(xy)z, yzx, y(zx), (yz)x, z(xy), yxz, zyx, (xyz)\}$, except for a permutation of alternatives (see Remarks 14 and 16). Let the number of individuals holding the orderings 1. $(xy)z$ 2. yzx 3. $y(zx)$ 4. $(yz)x$ 5. $z(xy)$ 6. yxz 7. zyx, 8. (xyz) be $n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8$ respectively. Double the occurrences of yzx, yxz, zyx; replace each occurrence of $(xy)z$ by $(xyz \wedge yxz)$, of $y(zx)$ by $(yzx \wedge yxz)$, of $(yz)x$ by $(yzx \wedge zyx)$, of $z(xy)$ by $(zxy \wedge zyx)$; and delete all occurrences of (xyz). Then we obtain:

$ordering$	$occurrences$	$ordering$	$occurrences$	$difference$
xyz	n_1	zyx	$n_4 + n_5 + 2n_7$	$d_1 = n_1 - n_4 - n_5 - 2n_7$
yzx	$2n_2 + n_3 + n_4$	xzy	0	$d_2 = 2n_2 + n_3 + n_4$
zxy	n_5	yxz	$n_1 + n_3 + 2n_6$	$d_3 = n_5 - n_1 - n_3 - 2n_6$

As occurrences of xzy are 0 it is not possible for the reduced form to consist of occurrences of three linear orderings of Group II. As it is not possible to have $[d_1 > 0 \wedge d_3 > 0]$, the reduced form cannot consist of occurrences of three linear orderings of Group I. This establishes that (b) of Theorem 1 cannot hold.
If $n = 2k+1, k \in \mathbb{N}$ and $n_8 = 0$, then (a) of Theorem 1 cannot hold; as $[d_1 > 0 \wedge d_2 > 0 \wedge d_3 = 0]$ is not possible; $[d_1 = 0 \wedge d_2 > 0 \wedge d_3 > 0]$ is not possible; and $[d_1 < 0 \wedge d_2 = 0 \wedge d_3 < 0 \wedge d_1 = d_3 \to n \text{ is even}]$. □

Theorem 4 *Let $n = 2k+1, k \in \mathbb{N}$. Let $\mathcal{D} \subseteq \mathcal{T}$. Then the method of majority decision f yields transitive social $R, R = f(R_1, \ldots, R_n)$, for every $(R_1, \ldots, R_n) \in \mathcal{D}^n$ iff $\mathcal{D}$ satisfies the condition of weak Latin Square partial agreement.*

Proof: Let $\mathcal{D} \subseteq \mathcal{T}$. Let $\mathcal{D}$ satisfy WLSPA. As WLSPA is the union of VR(1), TP, ER, transitivity under the MMD for every $(R_1, \ldots, R_n) \in \mathcal{D}^n$ follows from Propositions 3,4,1.

From the definition of WLSPA, it follows that if $\mathcal{D} \subseteq \mathcal{T}$ violates WLSPA then it must contain $[(xPyPz \wedge yRzPx \wedge zRxRy) \vee (xPyPz \wedge yRzRx \wedge zPxRy)]$, except for a formal interchange of alternatives.

Suppose $\mathcal{D}$ contains $(xPyPz \wedge yRzPx \wedge zRxRy)$. Consider a profile such that $n(xPyPz) = k, n(yRzPx) = k, n(zRxRy) = 1$. For the reduced form: k occurrences of xyz will result in $2k$ occurrences of xyz. If $yRzPx$ is $yPzPx$, then k occurrences of $yRzPx$ will result in $2k$ occurrences of yzx; and if $yRzPx$ is $yIzPx$, then k occurrences of $yRzPx$ will result in k occurrences of yzx and k occurrences of zyx. If $zRxRy$ is $zPxPy$, then 1 occurrence of $zRxRy$ will result in 2 occurrences of zxy; if $zRxRy$ is $zPxIy$, then 1 occurrence of $zRxRy$ will result in 1 occurrence of zxy and 1 occurrence of zyx; if $zRxRy$ is $zIxPy$, then 1 occurrence of $zRxRy$ will result in 1 occurrence of zxy and 1 occurrence of xzy; if $zRxRy$ is $zIxIy$, then 1 occurrence of $zRxRy$ will not result in any addition of orderings. Consequently, the reduced form will either consist of xyz, yzx, zxy with occurrences of $2k, 2k, 2$ respectively, or of $2k, 2k-1, 1$ respectively, or of $2k-1, 2k, 1$ respectively, or of $k, k, 2$ respectively, or of $k, k-1, 1$ respectively, or of $k-1, k, 1$ respectively; or of xyz, yzx with equal number of occurrences, equal occurrences being $2k$ or k; or of xyz, zxy with one occurrence each; or of yzx, zxy with with one occurrence each. Thus in all cases either (a) or (b) of Theorem 1 holds.

If $\mathcal{D}$ contains $[(xPyPz \wedge yRzRx \wedge zPxRy)$ then consider a profile such that $n(xPyPz) = k, n(yRzRx) = 1, n(zPxRy) = k$. Then the reduced form will either consist of xyz, yzx, zxy with occurrences of $2k, 2, 2k$ respectively, or of $2k, 1, 2k-1$ respectively, or of $2k-1, 1, 2k$ respectively, or of $k, 2, k$ respectively, or of $k, 1, k-1$ respectively, or of $k-1, 1, k$ respectively; or of xyz, zxy with equal number of occurrences, equal occurrences being $2k$ or k; or of xyz, yzx with one occurrence each; or of yzx, zxy with with one occurrence each. Thus, once again, in all cases either (a) or (b) of Theorem 1 holds. □

Theorem 5 *Let $n \geq 5$. Let $\mathcal{D} \subseteq \mathcal{T}$. Then the method of majority decision f yields quasi-transitive social $R, R = f(R_1, \ldots, R_n)$, for every $(R_1, \ldots, R_n) \in \mathcal{D}^n$ iff $\mathcal{D}$ satisfies the condition of Latin Square partial agreement.*

Proof: Let $\mathcal{D} \subseteq \mathcal{T}$. Let $\mathcal{D}$ satisfy LSPA. As LSPA is the union of VR(2), LA, ER, quasi-transitivity under the MMD for every $(R_1, \ldots, R_n) \in \mathcal{D}^n$ follows from Propositions 3,4,1.

From the definition of LSPA, it follows that if $\mathcal{D} \subseteq \mathcal{T}$ violates LSPA then it must contain one of the following 6 sets of orderings, except for a formal interchange of alternatives:

$$
\begin{array}{lll lll lll}
(I) & 1. & xPyPz & (II) & 1. & xPyPz & (III) & 1. & xPyPz \\
 & 2. & yPzPx & & 2. & yPzPx & & 2. & yPzPx \\
 & 3. & zPxPy & & 3. & zPxIy & & 3. & zIxPy \\
 & & & & & & & & \\
(IV) & 1. & xPyPz & (V) & 1. & xPyPz & (VI) & 1. & xPyPz \\
 & 2. & yPzIx & & 2. & yIzPx & & 2. & yIzPx \\
 & 3. & zPxIy & & 3. & zIxPy & & 3. & zPxIy
\end{array}
$$

Consider any $\mathcal{D}$ containing one of these six sets and let $(R_1, \ldots, R_n)$ be a profile such that the ordering 1 of the set is held by n_1 individuals, ordering 2 of the set is held by n_2 individuals, ordering 3 of the set is held by n_3 individuals, and $n_1 + n_2 + n_3 = n$. We consider each of these six cases.

(I) The reduced form consists of xyz, yzx, zxy with $2n_1, 2n_2, 2n_3$ occurrences respectively. Let (n_1, n_2, n_3) be (k, k, k) if $n = 3k,\ k \geq 2$; be $(k+1, k, k)$ if $n = 3k+1,\ k \geq 2$; be $(k+1, k+1, k)$ if $n = 3k+2,\ k \geq 1$. Then the occurrences of none of the three orderings exceeds half of the total number of occurrences. Thus (b) of Theorem 1 holds.

(II) The reduced form consists of xyz, yzx, zxy with $2n_1 - n_3, 2n_2, n_3$ occurrences respectively. Let (n_1, n_2, n_3) be $(k+1, k-1, k)$ if $n = 3k,\ k \geq 2$; be $(k+1, k, k)$ if $n = 3k+1$, $k \geq 2$; be $(k+1, k, k+1)$ if $n = 3k+2,\ k \geq 1$. Then the occurrences of none of the three orderings exceeds half of the total number of occurrences. Thus (b) of Theorem 1 holds.

(III) The reduced form consists of xyz, yzx, zxy with $2n_1, 2n_2 - n_3, n_3$ occurrences respectively. Let (n_1, n_2, n_3) be $(k-1, k+1, k)$ if $n = 3k,\ k \geq 2$; be $(k, k+1, k)$ if $n = 3k+1$, $k \geq 2$; be $(k, k+1, k+1)$ if $n = 3k+2,\ k \geq 1$. Then the occurrences of none of the three orderings exceeds half of the total number of occurrences. Thus (b) of Theorem 1 holds.

(IV) The reduced form consists of xyz, yzx, zxy with $2n_1 - n_3, n_2, n_3 - n_2$ occurrences respectively. Let (n_1, n_2, n_3) be $(k, k-1, k+1)$ if $n = 3k,\ k \geq 2$; be $(k, k, k+1)$ if $n = 3k+1,\ k \geq 2$; be $(k+1, k, k+1)$ if $n = 3k+2,\ k \geq 1$. Then the occurrences of none of the three orderings exceeds half of the total number of occurrences. Thus (b) of Theorem 1 holds.

(V) The reduced form consists of xyz, yzx, zxy with $2n_1 - n_2, n_2 - n_3, n_3$ occurrences respectively. Let (n_1, n_2, n_3) be $(k, k+1, k-1)$ if $n = 3k,\ k \geq 2$; be $(k, k+1, k)$ if $n = 3k+1$, $k \geq 2$; be $(k+1, k+1, k)$ if $n = 3k+2,\ k \geq 1$. Then the occurrences of none of the three orderings exceeds half of the total number of occurrences. Thus (b) of Theorem 1 holds.

(VI) The reduced form consists of xyz, yzx, zxy with $2n_1 - n_2 - n_3, n_2, n_3$ occurrences respectively. Let (n_1, n_2, n_3) be $(k+1, k, k-1)$ if $n = 3k,\ k \geq 2$; be $(k+1, k, k)$ if $n = 3k+1,\ k \geq 2$; be $(k+1, k+1, k)$ if $n = 3k+2,\ k \geq 1$. Then the occurrences of none of the three orderings exceeds half of the total number of occurrences. Thus (b) of Theorem 1 holds.

This establishes the theorem. $\square$

Theorem 6 *Let $n = 4$. Let $\mathcal{D} \subseteq \mathcal{T}$. Then the method of majority decision f yields quasi-transitive social $R, R = f(R_1, \ldots, R_4)$, for every $(R_1, R_2, R_3, R_4) \in \mathcal{D}^4$ iff $\mathcal{D}$ satisfies the condition of weak extremal restriction.*

Proof: Let $\mathcal{D} \subseteq \mathcal{T}$. Let $\mathcal{D}$ satisfy WER. In view of Theorem 5, it suffices to consider only those $\mathcal{D}$ which satisy WER but violate LSPA. As $n = 4$, we need consider only those $\mathcal{D}$ satisfying WER and violating LSPA which contain at most 4 orderings. Any such $\mathcal{D}$ must contain, except for a permutation of alternatives, $[xyz \wedge y(zx) \wedge z(xy)] \vee [xyz \wedge (yz)x \wedge (zx)y] \vee [xyz \wedge y(zx) \wedge z(xy) \wedge [xzy \vee zyx \vee yxz \vee x(yz) \vee (xy)z \vee (zx)y \vee (xyz)]] \vee [xyz \wedge (yz)x \wedge (zx)y \wedge [xzy \vee zyx \vee yxz \vee x(yz) \vee y(zx) \vee (xy)z \vee (xyz)]]$. The reduced form does not consist of occurrences of three orderings of the same group in any of the cases. Thus, (b) of Theorem 1 cannot hold.

From the definition of WER, it follows that if $\mathcal{D} \subseteq \mathcal{T}$ violates WER then it must contain one of the following 4 sets of orderings, except for a formal interchange of alternatives:

(I) 1. $xPyPz$		(II) 1. $xPyPz$	(III) 1. $xPyPz$	(IV) 1. $xPyPz$
2. $yPzPx$		2. $yPzPx$	2. $yIzPx$	2. $yIzPx$
3. $zPxPy$		3. $zPxIy$	3. $zPxPy$	3. $zPxIy$

For each set, consider $(R_1, \ldots, R_n)$ such that the ordering 1 of the set is held by 2 individuals, ordering 2 of the set is held by 1 individual, ordering 3 of the set is held by 1 individual. The reduced form, in each case, consists of occurrences of the three orderings of Group I, the occurrences of none of them exceeding half the total number of occurrences. Thus (b) of Theorem 1 holds. $\square$

Theorem 7 *Let $n = 3$. Let $\mathcal{D} \subseteq \mathcal{T}$. Then the method of majority decision f yields quasi-transitive social $R, R = f(R_1, R_2, R_3)$, for every $(R_1, R_2, R_3) \in \mathcal{D}^3$ iff $\mathcal{D}$ satisfies the condition of Latin Square linear ordering restriction.*

Proof: Let $\mathcal{D} \subseteq \mathcal{T}$. Let $\mathcal{D}$ satisfy LSLOR. In view of Theorem 6, it suffices to consider only those $\mathcal{D}$ which satisfy LSLOR but violate WER. As $n = 3$, we need consider only those $\mathcal{D}$ satisfying LSLOR and violating WER which contain at most 3 orderings. Any such $\mathcal{D}$ must contain, except for a permutation of alternatives, $[xyz \wedge (yz)x \wedge z(xy)]$. The reduced form of this contains only two orderings of Group I. Thus, (b) of Theorem 1 cannot hold.

From the definition of LSLOR, it follows that if $\mathcal{D} \subseteq \mathcal{T}$ violates LSLOR then it must contain one of the following 3 sets of orderings, except for a formal interchange of alternatives:

(I) 1. $xPyPz$	(II) 1. $xPyPz$	(III) 1. $xPyPz$
2. $yPzPx$	2. $yPzPx$	2. $yPzPx$
3. $zPxPy$	3. $zPxIy$	3. $zIxPy$

For each set, consider (R_1, R_2, R_3) such that the each of the three orderings is held by one individual. The reduced form, in each case, consists of occurrences of the three orderings of Group I, the occurrences of none of them exceeding half the total number of occurrences. Thus (b) of Theorem 1 holds. □

Theorem 8 *If $(R_1, \ldots, R_n)$ is such that the number of individuals having linear orderings of Group I is equal to the number of individuals having linear orderings of Group II, then under the MMD $(R_1, \ldots, R_n)$ yields transitive social R.*

Proof: Each occurrence of a non-linear concerned ordering contributes for the reduced form one linear ordering of Group I and one linear ordering of Group II, and the unconcerned ordering does not contribute any ordering to either of the two Groups. Consequently, the equality of the number of linear orderings of Group I and the number of linear orderings of Group II will hold for the reduced form as well. This implies that $d^1 + d^2 + d^3 = 0$. Therefore it is not possible for the reduced form to consist of occurrences of three orderings of the same Group; or of occurrences of just two orderings of the same group. Therefore neither (a) nor (b) of Theorem 1 can hold. □

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