Characterization of Monotonicity and Neutrality for Binary Paretian Social Decision Rules

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Abstract

Main result of the paper shows that a social decision rule satisfying the conditions of unrestricted domain, independence of irrelevant alternatives and Pareto-criterion is neutral and monotonic if and only if Pareto quasi-transitivity holds.

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In the context of binary social decision rules the properties of neutrality and monotonicity, perhaps two of the most discussed conditions in the social choice literature, seem to be closely related to the various rationality conditions and the Pareto-criterion. Arrow [1] showed that if a binary social decision rule (SDR) satisfying the conditions of unrestricted domain and the weak Pareto-criterion always yields transitive social weak preference relation then, if a group of individuals is almost decisive for some ordered pair of alternatives then it is decisive for every ordered pair of alternatives. In other words, satisfaction of transitivity implies both neutrality and monotonicity with respect to almost decisiveness relative to the entire society. Gibbard pointed out that this result holds even if transitivity requirement is weakened to quasi-transitivity. Guha [4] and Blau [2] have shown that if weak Pareto-criterion is replaced by the Pareto-criterion (also called the strict Pareto-criterion), then quasi-transitivity is sufficient to ensure neutrality and monotonicity everywhere, not just with respect to almost decisiveness relative to the entire society.

In this paper we derive a necessary and sufficient condition for a binary social decision rule satisfying the conditions of unrestricted domain and Pareto-criterion to be neutral and monotonic. It is shown that if an SDR satisfies the conditions of unrestricted domain, independence of irrelevant alternatives, and Pareto-criterion then it is neutral and monotonic if and only if it satisfies Pareto quasi-transitivity. Pareto quasi-transitivity requires that if x is judged to be socially better than y and y is Pareto-superior to z then x must be judged to be socially better than z, and if x is Pareto-superior to y and y is judged to be better than z by the society then x must be considered socially better than z. In the presence of Pareto-criterion, Pareto quasi-transitivity is a much weaker condition than quasi-transitivity. There is complete logical independence between Pareto quasi-transitivity and acyclicity with or without the Pareto-criterion.

We also obtain, for the class of binary social decision rules satisfying the condition of unrestricted domain, joint characterization for two properties which have played an important role in the context of Arrow impossibility theorem and related results, namely (i) whenever a group of individuals is almost decisive for some ordered pair of alternatives, it is decisive for every ordered pair of alternatives and (ii) whenever a group of individuals is almost semidecisive for some ordered pair of alternatives, it is semidecisive for every ordered pair of alternatives. We show that for a binary social decision rule with unrestricted domain the above mentioned properties (i) and (ii) hold if and only if weak Pareto quasi-transitivity is satisfied. The weak Pareto quasi-transitivity, based on the weak Pareto criterion, is a less stringent requirement than the Pareto quasi-transitivity. Weak Pareto quasi-transitivity requires that if x is judged to be better than y by the society and all individuals prefer y to z then x must be judged to be socially better than z, and if all individuals prefer x to y and y is regarded to be socially better than z then x must be judged to be socially better than z.

Definitions and Assumptions

The set of social alternatives S is assumed to contain at least 3 alternatives. Alternatives are defined in such a way that they are mutually exclusive. The set of individuals constituting the society would be denoted by N. The cardinality of N would be denoted by L, where L is finite and greater than one. Each individual $i \in N$ has a binary weak preference relation R_i (at least as good as) defined over S.

Corresponding to a binary relation R over S we define the relations P (better than) and I (indifferent to) in the usual way, i.e., $\forall\ x, y \in S$: $[(x\ P\ y \longleftrightarrow x\ R\ y \wedge \sim y\ R\ x) \wedge (x\ I\ y \longleftrightarrow x\ R\ y \wedge y\ R\ x)]$. We define a binary relation R over a set S to be (a) reflexive iff $\forall\ x \in S : x\ R\ x$ (b) connected iff $\forall\ x, y \in S : (x \neq y \longrightarrow x\ R\ y \vee y\ R\ x)$ (c) acyclic iff $\forall\ x_1, \ldots, x_j \in S : (x_1\ P\ x_2 \wedge x_2\ P\ x_3 \wedge \ldots \wedge x_{j-1}\ P\ x_j \longrightarrow x_1\ R\ x_j)$

(d) quasi-transitive iff $\forall x, y, z \in S : (x P y \wedge y P z \longrightarrow x P z)$ (e) transitive iff $\forall x, y, z \in S : (x R y \wedge y R z \longrightarrow x R z)$ (f) an ordering iff R is reflexive, connected and transitive. Throughout this paper we would assume that each R_i is an ordering. For $\forall x, y \in S$ we define, $x Q y$ iff $\forall i \in N : x R_i y \wedge \exists i \in N : x P_i y$, $x T y$ iff $\forall i \in N : x P_i y$, and $x J y$ iff $\forall i \in N : x I_i y$.

A social decision rule (SDR) f is a functional relation which for every ordered L - tuple $(R_1, \ldots, R_L)$, to be written $< R_i >$ in the abbreviated form, in its domain assigns a unique reflexive and connected social binary weak preference relation R. An SDR satisfies (a) the condition of unrestricted domain (U) iff every logically possible configuration of individual orderings is in the domain of the rule (b) the condition of independence of irrelevant alternatives (I) iff for all $< R_i >$ and $< R_i' >$ in the domain of f, if $< R_i >$ and $< R_i' >$ are identical over $A \subset S$ then R and R' are identical over A, where R and R' are social weak preference relations corresponding to $\langle R_i \rangle$ and $\langle R'_i \rangle$ respectively (c) the weak Pareto-criterion (P) iff $\forall x, y \in S : (x T y \longrightarrow x P y)$ (d) the Pareto-preference (P*) iff $\forall x, y \in S : (x Q y \longrightarrow x P y)$ (e) the Pareto-indifference (P**) iff $\forall x, y \in S :$ $(x J y \longrightarrow x I y)$ (f) the Pareto-criterion iff P* and P** hold (g) the weak Pareto quasi-transitivity (WPQT) iff $\forall x, y, z \in S : [(x P y \wedge y T z \longrightarrow x P z) \wedge (x T y \wedge y P z \longrightarrow x P z)]$ (h) the Pareto quasi-transitivity (PQT) iff $\forall x, y, z \in S : [(x P y \wedge y Q z \longrightarrow x P z) \wedge (x Q y \wedge y P z \longrightarrow x P z)]$. Both WPQT and PQT are conjunctions of two conditions each, the second parts of which were first defined by Fishburn [3].

A social decision rule satisfying condition I would be called a binary rule. A binary rule satisfies (a) neutrality (N) iff for all $\langle R_i \rangle$ and $\langle R'_i \rangle$ in the domain of the rule and $\forall x, y, z, w \in S :$

$$[[\forall i : [(x R_i y \longleftrightarrow z R'_i w) \wedge (y R_i x \longleftrightarrow w R'_i z)]] \longrightarrow [(x R y \longleftrightarrow z R' w) \wedge (y R x \longleftrightarrow w R' z)]]$$

(b) monotonicity (M) iff for all $\langle R_i \rangle$ and $\langle R'_i \rangle$

in the domain of the rule and $\forall x, y \in S$: $[\forall i : [(x P_i y \longrightarrow x P_i' y) \wedge (x I_i y \longrightarrow x R_i' y)] \longrightarrow [(x P y \longrightarrow x P' y) \wedge (x I y \longrightarrow x R' y)]]$

(c) weak monotonicity (WM) iff for all $\langle R_i \rangle$ and $\langle R_i' \rangle$ in the domain of the rule, $\forall x, y \in S$, and $\forall k \in N : [[\forall i \neq k : [(x R_i y \longleftrightarrow x R_i' y) \wedge (y R_i x \longleftrightarrow y R_i' x)] \wedge [(x P_k y \longrightarrow x P_k' y) \wedge (x I_k y \longrightarrow x R_k' y)]] \longrightarrow [(x P y \longrightarrow x P' y) \wedge (x I y \longrightarrow x R' y)]]$.

We define a set of individuals $V \subset N$ to be

(a) (N - A) - almost decisive for x against y $[D_A (x, y)]$ iff $(\forall i \in A : x I_i y \wedge \forall i \in V : x P_i y \wedge \forall i \in N - A - V : y P_i x) \longrightarrow x P y$, where $A \subsetneq N$

(b) (N - A) - decisive for x against y $[\bar{D}_A (x, y)]$ iff $(\forall i \in A : x I_i y \wedge \forall i \in V : x P_i y) \longrightarrow x P y$, where $A \subsetneq N$ (c) (N - A) - decisive iff it is (N - A) - decisive for every ordered pair of alternatives

(d) (N - A) - almost semidecisive for x against y $[S_A (x, y)]$ iff $(\forall i \in A : x I_i y \wedge \forall i \in V : x P_i y \wedge \forall i \in N - A - V : y P_i x) \longrightarrow x R y$, where $A \subsetneq N$

(e) (N - A) - semidecisive for x against y[$\bar{S}_A$ (x,y)] iff ($\forall i \in A : x \, I_i \, y \wedge \forall i \in V : x \, P_i \, y) \longrightarrow x \, R \, y$, where $A \subsetneq N$ (f) (N - A) - semidecisive iff it is (N - A) - semidecisive for every ordered pair of alternatives.

If $A = \phi$ then we drop the prefix (N - A). (a) through (f) then give definitions of an almost decisive set for x against y [D (x, y)], a decisive set for x against y [$\bar{D}$ (x, y)], a decisive set, an almost semidecisive set for x against y [S (x, y)], a semidecisive set for x against y [$\bar{S}$ (x, y)], and a semidecisive set respectively.

Characterization Theorems

Lemma 1 : Let social decision rule satisfy unrestricted domain, independence of irrelevant alternatives and weak Pareto quasi-transitivity. Then, if a group of individuals V is almost decisive for some ordered pair of distinct alternatives then it is decisive for every ordered pair of distinct alternatives.

Proof : Let V be almost decisive for (x, y), $x \neq y$. Consider the following configuration of preferences, where z is an alternative distinct from x and y:

$\forall i \in V : x P_i y P_i z$

$\forall i \in N - V : y P_i x \wedge y P_i z$

We obtain $x P y$ by the almost decisiveness of V for (x, y) and the fact that $\forall i \in V : x P_i y \wedge \forall i \in N - V : y P_i x$. $x P y$ and $\forall i \in N : y P_i z$ imply $x P z$ by WPQT. As the preferences of individuals belonging to $N - V$ have not been specified over $\{x, z\}$, it follows that V is decisive for (x, z). Similarly by considering the configuration $\forall i \in V : z P_i x P_i y$, $\forall i \in N - V : (y P_i x \wedge z P_i x)$ we can show that $D(x, y) \longrightarrow \bar{D}(z, y)$. By appropriate interchanges of alternatives it follows that $D(x, y)$ implies $\bar{D}(a, b)$ for all $(a, b) \in \{x, y, z\} X \{x, y, z\}$, where $a \neq b$. To prove the assertion for any $(a, b) \in S X S$, $a \neq b$, first we note that if $(a = x$ or $y)$ or $(b = x$ or $y)$ the desired conclusion can be obtained by considering a triple which includes all of x, y, a, and b. If both a and b are different from x and y then one first considers the triple $\{x, y, a\}$ and deduces $\bar{D}(x, a)$ and hence $D(x, a)$, and then considers the triple $\{x, a, b\}$ and obtains $\bar{D}(a, b)$.

Lemma 2 : Let social decision rule satisfy unrestricted domain, independence of irrelevant alternatives and weak Pareto quasi-transitivity. Then, if a group of individuals V is almost semidecisive for some ordered pair of distinct alternatives then it is semidecisive for every ordered pair of distinct alternatives.

Proof : Let V be almost semidecisive for (x, y), $x \neq y$. Let z be an alternative distinct from x and y and consider the following configuration of preferences :

$\forall i \in V : \; x P_i y P_i z$

$\forall i \in N - V : \; y P_i x \wedge y P_i z$

We obtain $x R y$ by the almost semidecisiveness of V for (x, y) and the fact that $\forall i \in V : x P_i y$ and $\forall i \in N-V : y P_i x$. Suppose $z P x$. $\forall i \in N : y P_i z$ and $z P x$ imply $y P x$ by WPQT, which contradicts $x R y$. Therefore $z P x$ cannot be true, which by the connectedness of R implies $x R z$. As the preferences of individuals belonging to N-V have not been specified over $\{x, z\}$, from the fact of $x R z$ we conclude $S(x, y) \longrightarrow \bar{S}(x, z)$. Similarly by considering the configuration $\forall i \in V$: $z P_i x P_i y$, $\forall i \in N-V$: $(y P_i x \wedge z P_i x)$ we can show that $S(x, y) \longrightarrow \bar{S}(z, y)$. By appropriate interchanges of alternatives it follows that $S(x, y)$ implies $\bar{S}(a, b)$ for all $(a, b) \in \{x, y, z\} X \{x, y, z\}$, where $a \neq b$. Now, the rest of the proof establishing that $S(x, y)$ implies $\bar{S}(a, b)$ for all $(a, b) \in S X S$, $a \neq b$, is exactly similar to that of lemma 1 and will be omitted here.

As weak Pareto quasi-transitivity is a weaker condition than the conjunction of weak Pareto-criterion and quasi-transitivity, lemma 1 generalizes the Arrow-Gibbard result and lemma 2 Sen's result.

Theorem 1 : If a binary social decision rule has unrestricted domain then (i) whenever a group of individuals is almost decisive for some ordered pair of distinct alternatives, it is decisive for every ordered pair of distinct alternatives and (ii) whenever a group of individuals is almost semidecisive for some ordered pair of distinct alternatives, it is semidecisive for every ordered pair of distinct alternatives, if and only if weak Pareto quasi-transitivity holds.

Proof : Let properties (i) and (ii) hold and suppose $x P y$ and $\forall i \in N : y P_i z$, where $x, y, z \in S$. Let the configuration of individual preferences over $\{x, y\}$ in this situation $\langle R_i \rangle$ be characterized as follows:

$\forall i \in N_1 : x P_i y$

$\forall i \in N_2 : x I_i y$

$\forall i \in N_3 : y P_i x$,

where $\bigcup_{i=1}^{3} N_i = N$.

As $\forall i \in N : y P_i z$, it follows that we must have $\forall i \in N_1 \cup N_2 : x P_i z$. Now consider the following configuration of individual preferences over $\{x, y, z\}$:

$\forall i \in N_1 : x P_i' y P_i' z$

$\forall i \in N_2 : x I_i' y P_i' z$

$\forall i \in N_3 : y P_i' z P_i' x$.

Suppose that $z R' x$ obtains. Then N_3 is almost semidecisive for (z, x) and therefore by property (ii) a semidecisive set for every ordered pair of alternatives. But then in the $\langle R_i \rangle$ situation we must have $y\ R\ x$ as all individuals belonging to N_3 prefer y to x. This, however, contradicts the hypothesis that $x\ P\ y$ obtains. Therefore $z\ R'\ x$ cannot be true. As R' is connected, it follows that $x\ P'\ z$ must hold. This implies that $N_1 \cup N_2$ is almost decisive for (x, z). By property (i) then $N_1 \cup N_2$ is a decisive set for every ordered pair of alternatives. Therefore in the $\langle R_i \rangle$ situation we must have $x\ P\ z$ as $\forall i \in N_1 \cup N_2$: $x\ P_i\ z$ obtains. Thus we have shown that $x\ P\ y$ and $\forall i \in N$: $y\ P_i\ z$ imply $x\ P\ z$. By an analogous argument it can be demonstrated that $\forall i \in N$: $x\ P_i\ y$ and $y\ P\ z$ also imply $x\ P\ z$. This establishes that properties (i) and (ii) imply WPQT. That WPQT implies properties (i) and (ii) has already been shown in lemmas 1 and 2.

Remark 1 : It is worth noting that properties (i) and (ii) are logically independent of each other, as can be seen by the following examples of social decision rules defined for $S = \{x, y, z\}$ and $N = \{1, 2, 3\}$:

(a) The method of majority decision satisfies both the properties.

(b) The imposed rule which always yields $R = x P y \wedge y I z \wedge x I z$ violates both the properties.

(c) The following SDR satisfies property (ii) but violates (i) :

$\forall a, b \in S : (\exists i : a I_i b \longrightarrow a I b)$

$\forall a, b \in S : [\sim (\exists i : a I_i b) \wedge ($ number of individuals with $a P_i b >$ number of individuals with $b P_i a) \longrightarrow a P b]$

(d) The following SDR satisfies property (i) but violates (ii) :

$x I y$, irrespective of individual preferences over $\{x, y\}$

$y I z$, irrespective of individual preferences over $\{y, z\}$

$(x I_1 z \wedge x P_2 z \wedge z P_3 x) \longrightarrow x P z$

$\sim (x I_1 z \wedge x P_2 z \wedge z P_3 x) \longrightarrow x I z$.

Lemma 3 : If a social decision rule satisfies unrestricted domain, independence of irrelevant alternatives and Pareto quasi-transitivity then, whenever a group of individuals is (N - A) - almost decisive for some ordered pair of distinct alternatives, it is (N - A) - decisive for every ordered pair of distinct alternatives, where $A \subsetneq N$.

Proof : Let V be (N - A) - almost decisive for (x, y), $x \neq y$. Let z be an alternative distinct from x and y , and consider the following configuration of preferences :

$\forall i \in A : x \, I_i \, y \, I_i \, z$

$\forall i \in V : x \, P_i \, y \, P_i \, z$

$\forall i \in N - A - V : y \, P_i \, x \wedge y \, P_i \, z$

In view of the (N - A) - almost decisiveness of V for (x, y) and the fact that ($\forall i \in A : x \, I_i \, y \wedge \forall i \in V : x \, P_i \, y \wedge \forall i \in N - A - V : y \, P_i \, x$), we obtain x P y. From x P y and ($\forall i \in N : y \, R_i \, z \wedge \forall i \in N - A : y \, P_i \, z$) we conclude x P z by Pareto quasi-transitivity, as $N - A \neq \phi$. As the preferences of individuals in N - A - V have not been specified over $\{x, z\}$, it follows that V is (N - A) - decisive for (x, z). Similarly by considering the configuration

$\forall i \in A :\ x\ I_i\ y\ I_i\ z$, $\forall i \in V :\ z\ P_i\ x\ P_i\ y$, $\forall i \in N - A - V :\ (y\ P_i\ x \wedge z\ P_i\ x)$, we can show that $D_A(x, y) \longrightarrow \bar{D}_A(z, y)$. By appropriate interchanges of alternatives it follows that $D_A(x, y) \longrightarrow \bar{D}_A(a, b)$ for all $(a, b) \in \{x, y, z\} X \{x, y, z\}$, where $a \neq b$. As the rest of the proof establishing that $D_A(x, y)$ implies $\bar{D}_A(a, b)$ for all $(a, b) \in S X S$, $a \neq b$, is similar to that of lemma 1 , it will be omitted here.

Lemma 4 : If a social decision rule satisfies unrestricted domain, independence of irrelevant alternatives and Pareto quasi-transitivity then, whenever a group of individuals is (N - A) - almost semidecisive for some ordered pair of distinct alternatives, it is (N - A) - semidecisive for every ordered pair of distinct alternatives, where $A \subsetneq N$.

Proof : Let V be (N - A) - almost semidecisive for (x, y), $x \neq y$, and z be an alternative different from x and y. Consider the following configuration of preferences :

$\forall i \in A :\ x\ I_i\ y\ I_i\ z$

$\forall i \in V :\ x\ P_i\ y\ P_i\ z$

$\forall i \in N - A - V :\ y\ P_i\ x \wedge y\ P_i\ z$

We obtain $x\ R\ y$ by the $(N - A)$ - almost semidecisiveness of V for (x, y) and the fact of $(\forall i \in A : x\ I_i\ y \wedge \forall i \in V : x\ P_i\ y \wedge \forall i \in N - A - V : y\ P_i\ x)$. Suppose $z\ P\ x$. Then from $(\forall i \in N : y\ R_i\ z \wedge \forall i \in N - A : y\ P_i\ z)$ and $z\ P\ x$ we can conclude $y\ P\ x$ by Pareto quasi-transitivity, as $N - A \neq \phi$. This however contradicts $x\ R\ y$. Therefore $z\ P\ x$ cannot be true and so $x\ R\ z$ must hold. As the preferences of individuals belonging to $N - A - V$ have not been specified over $\{x, z\}$, it follows that V is $(N - A)$ - semidecisive for (x, z). Similarly by considering the configuration $\forall i \in A : x\ I_i\ y\ I_i\ z$, $\forall i \in V : z\ P_i\ x\ P_i\ y$, $\forall i \in N - A - V : z\ P_i\ x \wedge y\ P_i\ x$ we can show that $S_A\ (x, y) \longrightarrow \bar{S}_A\ (z, y)$. By appropriate interchanges of alternatives it follows that $S_A\ (x, y)$ implies $\bar{S}_A\ (a, b)$ for all $(a, b) \in \{x, y, z\} \times \{x, y, z\}$, $a \neq b$. The rest of the proof being similar to that of lemma 1 is omitted.

Theorem 2 : If a social decision rule satisfies unrestricted domain, Pareto-criterion, independence of irrelevant alternatives and Pareto quasi-transitivity then it is neutral.

Proof : Consider any $\langle R_i \rangle$ and $\langle R_i' \rangle$ such that

$\forall i \in N_1 : x P_i y$ $\qquad$ $\forall i \in N_1 : z P_i' w$

$\forall i \in N_2 : x I_i y$ $\qquad$ $\forall i \in N_2 : z I_i' w$

$\forall i \in N_3 : y P_i x$ $\qquad$ $\forall i \in N_3 : w P_i' z$

where $\bigcup_{i=1}^{3} N_i = N$

If $N_1 \cup N_3 = \phi$ then $x I y$ and $z I' w$ follow from Pareto-criterion. Now suppose $N_1 \cup N_3 \neq \phi$.

Suppose $x P y$. Then N_1 is $(N - N_2)$ - almost decisive for (x, y). In view of lemma 3, it follows that N_1 is $(N - N_2)$ - decisive for every ordered pair of alternatives. Therefore $z P' w$ must hold as $\forall i \in N_2 : z I_i' w$ and $\forall i \in N_1 : z P_i' w$. We have shown that $x P y \longrightarrow z P' w$. By an analogous argument it can be shown that $z P' w \longrightarrow x P y$. So we have $x P y \longleftrightarrow z P' w$. New suppose $y P x$. Then N_3 is an $(N - N_2)$ - almost decisive set for (y, x) and hence an $(N - N_2)$ - decisive set as a consequence of lemma 3. Therefore we must have $w P' z$ as $\forall i \in N_2 : w I_i' z$ and $\forall i \in N_3 : w P_i' z$. So $y P x \longrightarrow w P' z$. By a similar argument one obtains

$w\ P'\ z \longrightarrow y\ P\ x$. Therefore we have $y\ P\ x \longleftrightarrow w\ P'\ z$. As $x\ P\ y \longleftrightarrow z\ P'\ w$ and $y\ P\ x \longleftrightarrow w\ P'\ z$, by the connectedness of social R it follows that $x\ I\ y \longleftrightarrow z\ I'\ w$. This establishes that the SDR is neutral.

Lemma 5 : If a social decision rule f satisfies unrestricted domain and independence of irrelevant alternatives then it is monotonic if and only if it is weakly monotonic.

Proof : By the definitions of M and WM, if f is monotonic then it is weakly monotonic. Now suppose f is weakly monotonic. Consider $\langle R_i \rangle$ and $\langle R_i' \rangle$ such that $\forall i \in N : [(x\ P_i\ y \longrightarrow x\ P_i'\ y) \wedge (x\ I_i\ y \longrightarrow x\ R_i'\ y)]$, and let $N = \{1, 2, \ldots, m\}$ be the set of individuals for whom $R_i \neq R_i'$. Rename $\langle R_i \rangle$ and $\langle R_i' \rangle$ as $\langle R_i^o \rangle$ and $\langle R_i^m \rangle$ respectively and construct $\langle R_i^t \rangle$, $t = 1, 2, \ldots, m-1$, as follows :

$\forall i \neq t : R_i^t = R_i^{t-1}$

$i = t : R_i^t = R_i'$.

By WM we obtain $[(x\ P^{t-1}\ y \longrightarrow x\ P^t\ y) \wedge (x\ I^{t-1}\ y \longrightarrow x\ R^t\ y)]$ for $t = 1, 2, \ldots, m-1$, which implies $[(x\ P\ y \longrightarrow x\ P'\ y) \wedge (x\ I\ y \longrightarrow x\ R'\ y)]$.

Theorem 3 : If a social decision rule satisfies unrestricted domain, Pareto-criterion, independence of irrelevant alternatives and Pareto quasi-transitivity then it is monotonic.

Proof : In view of lemma 5 it suffices to show that the SDR is weakly monotonic. Consider any $\langle R_i \rangle$ and $\langle R_i' \rangle$ such that for all $i \neq k$ the restriction of R_i' over $\{x, y\}$ is identical with the restriction of R_i over $\{x, y\}$, and $[(y P_k x \wedge x R_k' y) \vee (x I_k y \wedge x P_k' y)]$. Let the restriction of $\langle R_i \rangle$ over $\{x, y\}$ be characterized as follows:

$\forall i \in N_1 : x P_i y$

$\forall i \in N_2 : x I_i y$

$\forall i \in N_3 : y P_i x$,

where $\bigcup_{i=1}^{3} N_i = N$.

If $N_1 \cup N_3 = \phi$ then $x I y$ and $x P' y$ follow by condition $\bar{P}$. Now let $N_1 \cup N_3 \neq \phi$.

Suppose $x P y$. Then N_1 is $(N - N_2)$ - almost decisive for (x, y) and hence an $(N - N_2)$ - decisive set in view of lemma 3. If $k \in N_3$ then it follows that we must have $x P' y$. Now suppose $k \in N_2$. If $y R' x$ then N_3 is $(N - (N_2 - \{k\}))$ - almost semidecisive for (y, x) and hence an $(N - (N_2 - \{k\}))$ - semidecisive set as a consequence of lemma 4. As $\forall i \in N_3 : y P_i x$ and $\forall i \in N_2 - \{k\} : x I_i y$, it follows that we must have $y R x$. This, however, contradicts the hypothesis that $x P y$ holds. So $y R' x$ is impossible and therefore $x P' y$ must obtain.

Next suppose $x I y$. Then N_1 is an $(N - N_2)$ - semidecisive set in view of lemma 4. If $k \in N_3$ then it follows that we must have $x R' y$ as $\forall i \in N_1 :$ $x P_i' y$ and $\forall i \in N_2 : x I_i' y$. Suppose $k \in N_2$ and $y P' x$. $y P' x$ implies that N_3 is an $(N - (N_2 - \{k\}))$ - decisive set, in view of lemma 3. This implies that $y P x$ must obtain as $\forall i \in N_3 : y P_i x$ and $\forall i \in N_2 - \{k\} : y I_i x$, contradicting the hypothesis of $x I y$. So $y P' x$ is impossible and by the connectedness of R we conclude that $x R' y$ must hold. Thus we have shown that $(x P y \longrightarrow x P' y)$ and $(x I y \longrightarrow x R' y)$, which establishes that the SDR is monotonic.

Theorem 4 : If a social decision rule satisfies unrestricted domain, Pareto-criterion, independence of irrelevant alternatives, neutrality and monotonicity then it satisfies Pareto quasi-transitivity.

Proof : Suppose $x P y$ and $(\forall i \in N : y R_i z \wedge \exists i : y P_i z)$. Let $\langle R_i \rangle$ over $\{x, y\}$ be as follows : $\forall i \in N_1 : x P_i y$, $\forall i \in N_2 : x I_i y$, $\forall i \in N_3 : y P_i x$, where $N_1 \cup N_2 \cup N_3 = N$. Let N_1', N_2' and N_3' designate the sets $\{i : x P_i z\}$, $\{i : x I_i z\}$, and $\{i : z P_i x\}$ respectively. From $(\forall i \in N : y R_i z \wedge \exists i : y P_i z)$ we conclude that $N_1 \subset N_1'$ and $N_3' \subset N_3$. Let $\langle R_i' \rangle$ be any situation such that $\forall i \in N_1 : x P_i' z$, $\forall i \in N_2 : x I_i' z$ and $\forall i \in N_3 : z P_i' x$. As $x P y$, we conclude $x P' z$ by neutrality. $x P' z$ in turn implies $x P z$ in view of $N_1 \subset N_1'$ and $N_3' \subset N_3$, as a consequence of monotonicity. Thus we have shown that $x P y$ and $(\forall i \in N : y R_i z \wedge \exists i : y P_i z)$ imply $x P z$.

By an analogous argument it can be shown that

$(\forall i \in N : x R_i y \wedge \exists i : x P_i y)$ and $y P z$ imply $x P z$. This establishes that Pareto quasi-transitivity holds.

Combing theorems 2, 3 and 4 we obtain :

Theorem 5 : A social decision rule satisfying unrestricted domain, Pareto - criterion and independence of irrelevant alternatives is neutral and monotonic if and only if Pareto quasi-transitivity holds.

Remark 2 : Theorem 5 can be slightly generalized by replacing Pareto-criterion in the statement of the theorem by Pareto - indifference. This follows in view of the following considerations :

(a) In the proof of theorem 2 Pareto-criterion has been used only to conclude $x I y$ and $z I' w$ when $\forall i \in N : (x I_i y \wedge z I_i' w)$. For this inference Pareto - indifference suffices.

(b) Pareto quasi-transitivity implies that $\forall x, y \in S : (x Q y \longrightarrow x R y)$. Suppose not. Then for some $x, y \in S : (x Q y \wedge y P x)$, which would imply $y P y$, a contradiction.

(c) In the proof of theorem 3 Pareto - criterion has been used only to conclude $x\ I\ y$ and $x\ P'\ y$ when $\forall i \in N : x\ I_i\ y$, $\forall i \in N - \{k\} : x\ I_i'\ y$, and $x\ P_k'\ y$. Here also, Pareto-criterion is not essential. $x\ I\ y$ can be concluded from Pareto-indifference and $x\ R'\ y$ by (b) which is a consequence of PQT.

(d) In the proof of theorem 4 Pareto-criterion has not been used at all.

Therefore we conclude :

A social decision rule satisfying unrestricted domain, Pareto-indifference, and independence of irrelevant alternatives is neutral and monotonic if and only if Pareto quasi-transitivity holds.

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* This paper was presented at the Fifth World Congress of the Econometric Society held at Boston (U.S.A.) in August 1985.