Characterization of Non-Minority Rules

Satish K. Jain
Centre for Economic Studies and Planning
School of Social Sciences
Jawaharlal Nehru University
New Delhi 110067
India

Abstract

It is shown that a social decision rule is (a) p-non-minority rule, $\frac{1}{2} \leq p < 1$, iff it satisfies the conditions of (i) independence of irrelevant alternatives (ii) neutrality (iii) monotonicity (iv) weak Pareto-criterion (v) anonymity and (vi) its structure is such that a coalition is blocking iff it is strictly blocking; (b) simple non-minority rule iff it satisfies conditions (i) — (vi) and its structure is such that every proper superset of a blocking coalition is winning; (c) simple non-minority rule defined for an odd number of individuals iff it satisfies conditions (i) — (iii), (v), (vi) and its structure is such that a coalition is blocking iff it is winning.

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Selection of a social decision rule by the society for aggregating individual preferences into social preferences is logically tantamount to accepting by the society the set of value-judgements which characterize the social decision rule. Therefore for the purpose of selecting a social decision rule it is of some importance to know which social decision rules are characterized by which sets of value-judgements. For three important social decision rules namely the method of majority decision, the Pareto-extension rule and the Borda rule such characterizations have been obtained by May [1952], Sen [1970] and Young [1974] respectively. The purpose of this paper is to do a similar exercise for the class of non-minority rules. A social decision rule is defined to be p-non-minority rule, $\frac{1}{2} \leq p < 1$, iff an alternative x is declared to be socially better than another alternative y iff the number of individuals who prefer x to y is greater than p times the number of individuals constituting the society. If p is equal to $\frac{1}{2}$ then the rule is called simple non-minority rule.

We show that a social decision rule is p-non-minority rule, $\frac{1}{2} \leq p < 1$, iff it satisfies the conditions of (i) independence of irrelevant alternatives (ii) neutrality (iii) monotonicity (iv) weak Pareto-criterion (v) anonymity and (vi) its structure is such that a coalition is blocking iff it is strictly blocking. A social decision rule is the simple non-minority rule iff it satisfies conditions (i) - (vi) mentioned above and its structure is such that every proper superset of a blocking coalition is winning. We also show that a social decision rule is the simple non-minority rule defined for an odd number of individuals iff it satisfies conditions (i) - (iii), (v), (vi) and its structure is such that a coalition is blocking iff it is winning.

1. Notation and Definitions

The set of social alternatives is denoted by S. S is assumed to contain at least two alternatives. We denote by N the finite set of individuals. It is assumed that # N = n $\geq$ 2. We assume that every individual i $\in$ N has a binary relation R_i on S. The asymmetric parts of binary relations R_i, R'_i, R, R$'$ etc. are denoted by P_i, P'_i, P, P$'$ etc. respectively; and symmetric parts by I_i, I'_i, I, I$'$ etc. respectively.

We define a binary relation R on a set S to be (i) reflexive iff $(\forall x \in S)(xRx)$, (ii) connected iff $(\forall x,y \in S)[x \neq y \rightarrow xRy \vee yRx]$, (iii) transitive iff $(\forall x,y,z \in S)[xRy \wedge yRz \rightarrow xRz]$, and (iv) an ordering iff it is reflexive, connected and transitive.

Throughout this paper it is assumed that for every individual i $\in$ N, R_i is an ordering.

We denote by C the set of all reflexive and connected binary relations on S and by T the set of all orderings on S. A profile of individual orderings $(R_1,......,R_n)$ specifies one and only one ordering of S for each individual i $\in$ N; $(R_1,....,R_n) : N \rightarrow T$. A social decision rule (SDR) f is a function which for every profile of individual orderings $(R_1,......,R_n) \in T^n$ determines a unique reflexive and connected social R, i.e., $f : T^n \rightarrow C$. Profiles $(R_1,.....,R_n)$, $(R'_1,......,R'_n)$ etc., will be written as $<R_i>$, $<R'_i>$ etc. respectively, in abbreviated form. The social binary relations corresponding to $<R_i>$, $<R'_i>$ etc. will be denoted by R, R$'$ etc. respectively. N ( ) will denote the number of individuals having the preferences specified in the parentheses.

An SDR satisfies the condition of independence of irrelevant alternatives (I) iff $(\forall <R_i>, <R'_i> \in T^n)$ $(\forall x, y \in S)$ $[(\forall i \in N)\ [(xR_iy \Leftrightarrow xR'_iy) \wedge (yR_ix \Leftrightarrow yR'_ix)] \rightarrow [(xRy \Leftrightarrow xR'y) \wedge (yRx \Leftrightarrow yR'x)]]$. An SDR satisfying the condition I satisfies (i) neutrality (N) iff $(\forall <R_i>, <R'_i> \in T^n)$ $(\forall x, y, z, w \in S)$ $[(\forall i \in N)$ $[(xR_iy \Leftrightarrow zR'_iw) \wedge (yR_ix \Leftrightarrow wR'_iz)] \rightarrow [(xRy \Leftrightarrow zR'w) \wedge (yRx \Leftrightarrow wR'z)]]$, and (ii) monotonicity (M) iff $(\forall <R_i>, <R'_i> \in T^n)$ $(\forall x, y \in S)$ $[(\forall i \in N)\ [(xP_iy \rightarrow xP'_iy) \wedge (xI_iy \rightarrow xR'_iy)] \rightarrow [(xPy \rightarrow xP'y) \wedge (xIy \rightarrow xR'y)]]$.

Let π denote the set of all permutations of positive integers 1, 2,....., n. An SDR satisfies the condition of (i) anonymity (A) iff $(\forall <R_i> \in T^n)$ $[(\exists \theta \in \pi)\ [<R'_i> = <R_{\theta(i)}>] \rightarrow R' = R]$, and (ii) weak Pareto-criterion (P) iff $(\forall <R_i> \in T^n)$ $(\forall x,y \in S)$ $[(\forall i \in N)(x P_i y) \rightarrow xPy]$.

A coalition is a subset of N. A coalition V is defined to be winning iff $(\forall <R_i> \in T^n)$ $(\forall x,y \in S)$ $[(\forall i \in V)(xP_iy) \rightarrow xPy]$. We denote by W the set of all winning coalitions. We define a coalition to be blocking iff ($\forall$ <R_i> $\in T^n$) ($\forall$x,y $\in$ S) [($\forall$i $\in$ V) (xP_iy) $\rightarrow$ xRy], and to be strictly blocking iff ($\forall$<R_i> $\in$ T^n) ($\forall$x,y $\in$ S) [($\forall$i $\in$ V) (xR_iy) $\rightarrow$ xRy]. The set of all blocking coalitions is denoted by B, and the set of all strictly blocking coalitions by B_s.

Remark 1 : Let f: $T^n \rightarrow$ C. If V_1, $V_2 \in$ W then $V_1 \cap V_2$ must be nonempty, because $V_1 \cap V_2 = \emptyset$ would lead to a contradiction if we have for x, y $\in$ S, [($\forall$i $\in V_1$) (xP_iy) $\wedge$ ($\forall$i $\in V_2$) (yP_ix)], which would imply (xPy $\wedge$ yPx).

Remark 2 : From the definitions of winning coalition, blocking coalition and strictly blocking coalition, it follows that if a coalition is winning or strictly blocking then it is blocking.

We define a social decision rule f : $T^n \rightarrow$ C to be p-non-minority rule, p $\in$ [$\frac{1}{2}$, 1), iff ($\forall$< R_i> $\in T^n$) ($\forall$x, y $\in$ S) [xPy $\Leftrightarrow$ N (xP_iy) > pn].

If p = $\frac{1}{2}$, the rule is called simple non-minority rule.

Remark 3 : From the definition of p-non-minority rule it is clear that a coalition is winning iff it contains more than pn individuals.

For any real number x we denote the largest integer less than or equal to x by [x].

2. Characterization Theorems

Theorem 1: A social decision rule f is p-non-minority rule, $\frac{1}{2} \leq p < 1$, iff it satisfies the conditions of (i) independence of irrelevant alternatives (ii) neutrality (iii) monotonicity (iv) weak Pareto-criterion (v) anonymity and (vi) its structure is such that a coalition is blocking iff it is strictly blocking.

Proof : Let f be p-non-minority rule, $\frac{1}{2} \leq p < 1$. Then from the definition of p-non-minority rule , $\frac{1}{2} \leq p < 1$, it is clear that f satisfies conditions (i)-(v). Suppose V, V $\subseteq$ N , is a blocking coalition . Then from the definitions of a blocking coalition and p-non-minority rule, $\frac{1}{2} \leq p < 1$, it follows that # (N-V) $\leq$ pn. Therefore, for any <R_i> $\in T^n$ and any x, y $\in$ S, ($\forall$i $\in$ V) (xR_iy) implies that N(yP_ix) $\leq$ pn, which in turn implies $\sim$ (yPx), i.e., xRy. This establishes that V is a strictly blocking coalition, thus proving that (vi) holds.

Next let social decision rule f satisfy conditions (i)-(vi). As f satisfies condition I we conclude that for any x, y $\in$ S the social R over {x,y} is completely determined by individual preferences over {x,y}. By neutrality the rule for determining social R from individual preferences is the same for all ordered pairs of alternatives. Consider any x, y $\in$ S and any profile <R_i> $\in T^n$ such that xPy. Let N_1, N_2, N_3 designate the sets {i $\in$ N | xP_iy}, {i $\in$ N | xI_iy}, {i $\in$ N | yP_ix} respectively. Now consider any profile <R_i'> $\in T^n$ such that [($\forall$i $\in N_1$) (xP_i'y) $\wedge$ ($\forall$i $\in N_2 \cup N_3$) (yP_i'x)]. Suppose yR$'$x. Then $N_2 \cup N_3$ is a blocking coalition as a consequence of conditions I, M and N. As every blocking coalition is strictly blocking we conclude that $N_2 \cup N_3$ is strictly blocking. But then in <R_i> situation we must have yRx, as we have ($\forall$i $\in N_2 \cup N_3$) (yR_ix). As this contradicts xPy, we conclude that in <R_i'> situation yR$'$x is impossible, i.e., we must have xP$'$y. xP$'$y in turn implies, in view of conditions I, M and N, that N_1 is a winning coalition. Thus we have shown that ($\forall$x, y $\in$ S) ($\forall$<R_i> $\in T^n$) [xPy $\rightarrow$ ($\exists$V $\in$ W) ($\forall$i $\in$ V) (xP_iy)]. If V $\in$ W then ($\forall$x,y $\in$ S) ($\forall$<R_i> $\in T^n$) [($\forall$i $\in$ V) (xP_iy) $\rightarrow$ xPy], by the definition of a winning coalition. Thus, ($\forall$x,y $\in$ S) ($\forall$<R_i> $\in T^n$) [xPy $\Leftrightarrow$ ($\exists$V $\in$ W) ($\forall$i $\in$ V) (xP_iy)].

Now, by the weak Pareto-criterion, N is winning and thus W is nonempty. If V $\in$ W and # V = k, then by anonymity and the definition of a winning coalition we conclude that ($\forall$V$'$ $\subseteq$ N) [#V$'$ $\geq$ k $\rightarrow$ V$'$ $\in$ W]. Next we note that ($\forall$ V $\subseteq$ N) [V $\in$ W $\rightarrow$ #V > $\frac{n}{2}$], otherwise, as a consequence of anonymity, there will exist two nonempty disjoint winning coalitions leading to a contradiction (see Remark 1). Let $\bar{k}$ = min { k | ($\exists$V $\in$ W) (# V = k) }. As $\bar{k} > \frac{n}{2}$, we obtain $\bar{k}$ = [pn] + 1 for some p $\in$ [$\frac{1}{2}$, 1). Therefore, we conclude that ($\exists$p $\in$ [$\frac{1}{2}$, 1)) ($\forall$ V $\subseteq$ N) [V $\in$ W $\Leftrightarrow$ # V > pn]. This coupled with the earlier inference that ($\forall$x,y $\in$ S) ($\forall$<R_i> $\in T^n$) [xPy $\Leftrightarrow$ ($\exists$V $\in$ W) ($\forall$i $\in$ V) (xP_iy)] implies that ($\exists$ p $\in$ [$\frac{1}{2}$,1)) ($\forall$x,y $\in$ S) ($\forall$ < R_i > $\in T^n$) [xPy $\Leftrightarrow$ N (xP_iy) > pn]. This establishes that f is p-non-minority rule, $\frac{1}{2} \leq p < 1$.

Theorem 2 : A social decision rule f is the simple non-minority rule iff it satisfies the conditions of (i) independence of irrelevant alternatives (ii) neutrality (iii) monotonicity (iv) weak Pareto-criterion (v) anonymity and its structure is such that (vi) a coalition is blocking iff it is strictly blocking and (vii) every proper superset of a blocking coalition is winning.

Proof : Let social decision rule f be the simple non-minority rule. Then, by Theorem 1, f satisfies conditions (i) - (vi). As every coalition which has more than $\frac{n}{2}$ individuals is winning, it follows that no coalition which has less than $\frac{n}{2}$ individuals can be blocking. Thus every blocking coalition has at least $\frac{n}{2}$ individuals. Consequently every proper superset of a blocking coalition has more than $\frac{n}{2}$ individuals and is thus winning. Thus (vii) holds.

Next let social decision rule f satisfy conditions (i)-(vii). Conditions (i)-(vi) imply that f must be p-non-minority rule for some $p \in [\frac{1}{2},1)$, by Theorem 1. Conditions I, M and N imply that if a coalition is not winning then its complement must be blocking. Let $\bar{k} = \min \{k \mid (\exists V \in W) (\# V = k)\}$. Then, it follows that every coalition which contains at least n - $\bar{k}$ + 1 individuals is blocking. As every proper superset of a blocking coalition is winning, it follows that every coalition which contains at least n - $\bar{k}$ + 2 individuals must be winning. Therefore, from the definition of $\bar{k}$ we conclude that :

$$n - \bar{k} + 2 \geq \bar{k}$$
$$\text{or} \qquad \bar{k} \leq \frac{n}{2} + 1$$

Also $\bar{k} > \frac{n}{2}$, otherwise by anonymity the existence of two nonempty disjoint winning coalitions would be implied leading to a contradiction. Thus we have $\frac{n}{2} < \bar{k} \leq \frac{n}{2} + 1$. This establishes that f is the simple non-minority rule.

Theorem 3 : A social decision rule f is the simple non-minority rule defined for an odd number of individuals iff f satisfies (i) independence of irrelevant alternatives (ii) neutrality (iii) monotonicity (iv) anonymity and its structure is such that (v) a coalition is blocking iff it is strictly blocking and (vi) a coalition is blocking iff it is winning.

Proof : let f be the simple non-minority rule with n an odd positive integer. Then, by Theorem 2, f satisfies conditions (i)-(v). From the definition of simple non-minority rule it follows that a coalition is blocking iff its complement is not winning, and that a coalition is winning iff it has more than $\frac{n}{2}$ individuals. Consequently $(\forall V \subseteq N) [V \in B \Leftrightarrow \# V \geq \frac{n}{2}]$. As n is odd, $(\forall V \subseteq N) [\# V \geq \frac{n}{2} \Leftrightarrow \# V > \frac{n}{2}]$. Therefore, $(\forall V \subseteq N)$ $[V \in B \Leftrightarrow V \in W]$, which establishes that (vi) holds.

Next suppose that social decision rule f satisfies (i)-(vi). Conditions I and N imply that $(\forall x, y \in S)$ $(\forall \langle R_i \rangle \in T^n)$ $[(\forall i \in N) (xI_iy) \rightarrow xIy]$. By conditions I, N and M then we can conclude that $(\forall x, y \in S)$ $(\forall \langle R_i \rangle \in T^n)$ $[(\forall i \in N) (xP_iy) \rightarrow xRy]$. This means that N is a blocking coalition and therefore by condition (vi) a winning coalition. Thus f satisfies the weak Pareto-criterion. As every blocking coalition is winning, f trivially satisfies the condition that every proper superset of a blocking coalition is winning. Thus f satisfies all the conditions of Theorem 2 and consequently f must be the simple non-minority rule. Now suppose n is even. Let V be a coalition such that $\# V = \frac{n}{2}$. Then, as V is not winning we conclude that $N - V$ must be blocking. But then by condition (vi) it follows that $N - V$ is winning. But as $\# (N - V) = \frac{n}{2}$, $N - V$ cannot be winning. This contradiction establishes that n must be odd, completing the proof of the theorem.

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