Non-Minority Rules : Characterization of Configurations with Rational Social Preferences

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Abstract

It is shown that for every non-minority rule a necessary and sufficient condition (i) for quasi-transitivity is that value-restriction or weakly conflictive preferences or unique-value restriction holds over every triple of alternatives (ii) for transitivity is that conflictive preferences or extreme-value restriction holds over every triple of alternatives.

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The purpose of this paper is to derive necessary and sufficient conditions for quasi-transitivity and transitivity of non-minority rules. One member of this class, namely the simple non-minority rule, also known as absolute (strict) majority rule, has been widely discussed in the literature. Several conditions on configurations of individual preferences have been formulated for the rationality of the social preference relation generated by the simple non-minority rule. Dummett and Farquharson [2] have shown that if in every triple of alternatives there exists an alternative which no individual regards as uniquely worst then the simple non-minority rule yields acyclic social preferences. Pattanaik [6] showed that the existence of an alternative in every triple which is regarded by none as uniquely best also guarantees acyclicity. In [3] Fine has derived necessary and sufficient conditions for the transitivity of the social preference relation.

We show that for every non-minority rule a necessary and sufficient condition for quasi-transitivity of the social preference relation is that the configuration of individual preferences satisfies, over every triple of alternatives, value-restriction (VR) or weakly conflictive preferences (WCP) or unique-value restriction (UVR). For every non-minority rule, satisfaction of extreme-value restriction (EVR) or conflictive preferences (CP) over every triple of alternatives is shown to be both necessary and sufficient for transitivity of the social preference relation. Of the four restrictions introduced in this paper, WCP and CP are partial antagonism conditions while UVR and EVR are in the same spirit as Sen's extremal restriction.

The interesting feature that emerges is that the necessary and sufficient conditions for quasi-transitivity or transitivity are same for all non-minority rules. This is in sharp contrast to the case of majority rules where conditions for transitivity are known to be different.

While extremal restriction is both necessary and sufficient for transitivity of the social preference relation generated by the simple majority rule, it is not sufficient for transitivity of the social preference relation generated by the two-thirds majority rule.

Restrictions on Preferences:

The set of social alternatives would be denoted by S. The cardinality n of S would be assumed to be finite and greater than 2. The set of individuals and the number of individuals are designated by L and N respectively. N ( ) would stand for the number of individuals holding the preferences specified in the parentheses and N_k for the number of individuals holding the k-th preference ordering. Each individual $i \in L$ is assumed to have an ordering R_i defined over S. The symmetric and asymmetric parts of R_i are denoted by I_i and P_i respectively. The social preference relation is denoted by R and its symmetric and asymmetric components by I and P respectively.

Non-Minority Rules : $\forall x, y \in S : x \; R \; y$ iff $N (y \; P_i \; x) \leqslant p N$, where p is a fraction such that $\frac{1}{2} \leqslant p < 1$. For $p = \frac{1}{2}$ we obtain the familiar simple non-minority rule.

An individual is defined to be concerned with respect to a triple iff he is not indifferent over every pair of alternatives belonging to the triple; otherwise he is unconcerned. For individual i, in the triple $\{x, y, z\}$, x is best iff $(x \; R_i \; y \wedge x \; R_i \; z)$; medium iff $(y \; R_i \; x \; R_i \; z \; \vee \; z \; R_i \; x \; R_i \; y)$; worst iff $(y \; R_i \; x \wedge z \; R_i \; x)$; uniquely best iff $(x \; P_i \; y \wedge x \; P_i \; z)$; uniquely medium iff $(y \; P_i \; x \; P_i \; z \; \vee \; z \; P_i \; x \; P_i \; y)$; and uniquely worst iff $(y \; P_i \; x \wedge z \; P_i \; x)$. Now, we define several restrictions which specify the permissible sets of individual orderings. All these restrictions are defined over triples of alternatives.

Value - Restriction (VR) : VR holds over a triple iff there is an alternative in the triple such that all concerned individuals agree that it is not best or all concerned individuals agree that it is not medium or all concerned individuals agree that it is not worst.

Weakly Conflictive Preferences (WCP) : Whenever an individual considers an alternative best in some strong ordering as worst, he regards the alternative worst in the strong ordering as best ; or alternatively whenever an individual considers an alternative worst in some strong ordering as best, he regards the alternative best in the strong ordering as worst. Formally, WCP holds over $\{x, y, z\}$ iff $[\forall a,b,c \in \{x, y, z\} : [\exists i: (a P_i b P_i c) \longrightarrow \forall i : ((b R_i a \wedge c R_i a) \longrightarrow c R_i b)]]$ $\vee [\forall a,b,c \in \{x, y, z\} : [\exists i : (a P_i b P_i c) \longrightarrow \forall i : ((c R_i a \wedge c R_i b) \longrightarrow b R_i a)]]$.

Unique - Value Restriction (UVR) : There exist distinct alternatives a and b in the triple such that a is not uniquely medium in any R_i , b is not uniquely best in any R_i , and whenever b is best in an R_i a is worst in that R_i , or alternatively there exist distinct a and b in the triple such that a is not uniquely medium in any R_i, b is not uniquely worst in any R_i, and whenever b is worst in an R_i a is best in that R_i. More formally, UVR holds over $\{x, y, z\}$ iff

$$[\ \exists \text{ distinct } a, b, c \in \{x, y, z\} : \forall i : [((a R_i b \wedge a R_i c) \vee (b R_i a \wedge c R_i a)) \wedge (a R_i b \vee c R_i b) \wedge (b R_i a \wedge b R_i c \longrightarrow c R_i a)]] \vee [\ \exists \text{ distinct } a, b, c \in \{x, y, z\} : \forall i : [((a R_i b \wedge a R_i c) \vee (b R_i a \wedge c R_i a)) \wedge (b R_i a \vee b R_i c) \wedge (a R_i b \wedge c R_i b \longrightarrow a R_i c)]].$$

Extreme - Value Restriction (EVR) : If an alternative is uniquely best in some ordering then in no ordering can it be medium unless it is worst also ; or alternatively if an alternative is uniquely worst in some ordering then in no ordering can it be medium unless it is best also, i.e., EVR holds over the triple $\{x, y, z\}$ iff $[\forall a, b, c \in \{x, y, z\} : [\exists i : (a P_i b \wedge a P_i c) \longrightarrow \forall i : [(b R_i a R_i c \longrightarrow c R_i a) \wedge (c R_i a R_i b \longrightarrow b R_i a)]]] \vee [\forall a, b, c \in \{x, y, z\} : [\exists i : (b P_i a \wedge c P_i a) \longrightarrow \forall i : [(b R_i a R_i c \longrightarrow a R_i b) \wedge (c R_i a R_i b \longrightarrow a R_i c)]]]$.

Conflictive Preferences (CP) : A set of individual orderings satisfies CP over the triple $\{x, y, z\}$

iff (i) there exists a partition of L_c into L_1 and L_2, where L_c is the set of individuals concerned with respect to $\{x, y, z\}$, such that $\forall i \in L_1$ have the same R - ordering, say, $x R_i y R_i z$ and $\forall i \in L_2$ have the opposite R - ordering $z R_i y R_i x$ and (ii) $\forall i \in L_1$ consider x to be uniquely best and $\forall i \in L_2$ consider x to be uniquely worst.

Lemma 1 : Conditions of value-restriction, weakly conflictive preferences and unique-value restriction are logically independent of each other.

Proof : The following 8 examples constitute a proof of complete logical independence of VR, WCP, and UVR.

(1) $x P_i y P_i z$
$x P_i z P_i y$

All three restrictions are satisfied.

(2) $y P_i z P_i x$
$z P_i y P_i x$
$y P_i x I_i z$
$z P_i x I_i y$

VR and WCP are satisfied and UVR is violated.

(3) $x \, P_i \, y \, P_i \, z$

$y \, P_i \, z \, P_i \, x$

VR and UVR are satisfied but WCP is violated.

(4) $y \, P_i \, x \, P_i \, z$

$y \, P_i \, z \, P_i \, x$

$z \, P_i \, x \, P_i \, y$

$z \, P_i \, y \, P_i \, x$

VR is satisfied and both WCP and UVR are violated.

(5) $x \, P_i \, y \, P_i \, z$

$y \, I_i \, z \, P_i \, x$

$z \, I_i \, x \, P_i \, y$

Both WCP and UVR are satisfied and VR is violated.

(6) $y \, I_i \, z \, P_i \, x$

$z \, I_i \, x \, P_i \, y$

$x \, I_i \, y \, P_i \, z$

VR and UVR are violated and WCP is satisfied.

(7) $x \, P_i \, y \, P_i \, z$

$y \, P_i \, z \, P_i \, x$

$x \, P_i \, z \, P_i \, y$

$z \, I_i \, y \, P_i \, x$

$y \, I_i \, x \, P_i \, z$

UVR is satisfied and VR and WCP are violated.

(8) $x \; P_i \; y \; P_i \; z$

$y \; P_i \; z \; P_i \; x$

$z \; P_i \; x \; P_i \; y$

All three restrictions are violated.

Lemma 2 : Extreme-value restriction and conflictive preferences conditions are logically independent of each other.

Proof : The proof consists of the following 4 examples:

(1) $x \; P_i \; y \; P_i \; z$

$z \; P_i \; y \; P_i \; x$

Both EVR and CP are satisfied.

(2) $x \; P_i \; y \; P_i \; z$

$z \; P_i \; y \; P_i \; x$

$x \; P_i \; y \; I_i \; z$

$z \; I_i \; y \; P_i \; x$

CP is satisfied and EVR is violated.

(3) $x \, P_i \, y \, P_i \, z$

$x \, P_i \, z \, P_i \, y$

EVR is satisfied but CP is violated.

(4) $x \, P_i \, y \, I_i \, z$

$x \, I_i \, y \, P_i \, z$

Both CP and EVR are violated.

Necessary and Sufficient Conditions for Quasi-Transitivity

Lemma 3 : A set of individual orderings violates all three restrictions VR, WCP and UVR over a triple $\{x, y, z\}$ iff it contains one of the following four 3-ordering sets, except for a formal interchange of alternatives ;

(A) $x \, P_i \, y \, P_i \, z$
$y \, P_i \, z \, P_i \, x$
$z \, P_i \, x \, P_i \, y$

(B) $x \, P_i \, y \, P_i \, z$
$y \, P_i \, z \, P_i \, x$
$z \, P_i \, x \, I_i \, y$

(C) $x \, P_i \, y \, P_i \, z$
$y \, P_i \, z \, P_i \, x$
$z \, I_i \, x \, P_i \, y$

(D) $x \, P_i \, y \, P_i \, z$
$y \, P_i \, z \, I_i \, x$
$z \, I_i \, x \, P_i \, y$

Proof : From the definition of WCP it follows that it is violated iff the set of orderings contains one of the following four sets, except for a formal interchange of alternatives,

(i) $x \, P_i \, y \, P_i \, z$
$y \, P_i \, z \, I_i \, x$
$z \, I_i \, x \, P_i \, y$

(ii) $x \, P_i \, y \, P_i \, z$
$y \, P_i \, z \, P_i \, x$

(iii) $x \, P_i \, y \, P_i \, z$
$x \, P_i \, z \, P_i \, y$
$z \, I_i \, x \, P_i \, y$
$z \, P_i \, y \, I_i \, x$

(iv) $x \, P_i \, y \, P_i \, z$
$y \, P_i \, x \, P_i \, z$
$z \, I_i \, x \, P_i \, y$
$x \, P_i \, z \, I_i \, y$

(i) It is the same set as D.

(ii) This configuration does not violate either VR or UVR. To violate UVR we have to include [$y \, P_i \, x \, P_i \, z \lor z \, P_i \, x \, P_i \, y \lor (z \, P_i \, y \, P_i \, x \land x \, P_i \, z \, P_i \, y) \lor x \, I_i \, z \, P_i \, y \lor z \, P_i \, x \, I_i \, y$] . Excepting the cases when we include $y \, P_i \, x \, P_i \, z$ or $(z \, P_i \, y \, P_i \, x \land x \, P_i \, z \, P_i \, y)$, in all other cases VR is also violated and it is seen that the set of R_i includes one of the four sets (A) - (D). To violate VR, if $y P_i x P_i z$ is included for violating UVR, we have to include $[$ concerned $R_i : z R_i x R_i y \vee ($ concerned $R_i : x R_i z R_i y \wedge$ concerned $R_i : z R_i y R_i x)]$, and in case $(z P_i y P_i x \wedge x P_i z P_i y)$ is included for violation of UVR, (concerned $R_i : z R_i x R_i y \vee$ concerned $R_i : y R_i x R_i z$) has to be included. With the required inclusion the set of R_i contains one of the four sets (A) - (D).

(iii) Neither VR nor UVR is violated. VR would be violated iff a concerned ordering in which y is best is included. Excepting the case when we include $y I_i z P_i x$, in all other cases UVR is also violated and one of the four sets (A) - (D) is contained in the set of R_i. In the case of inclusion of $y I_i z P_i x$, UVR is violated iff we include $[y P_i x P_i z \vee z P_i x P_i y \vee y P_i z P_i x \vee y P_i z I_i x \vee y I_i x P_i z]$.

In all cases one of (A) - (D) is contained in the set of R_i.

(iv) Again, neither VR nor UVR is violated. VR would be violated iff a concerned ordering in which x is worst is included. In all cases other than the case of inclusion of $z P_i y I_i x$, UVR is also violated and one of (A) - (D) is contained in the set of R_i. In the case of inclusion of $z P_i y I_i x$, UVR is violated iff we include $[\, x P_i z P_i y \;\; v \;\; y P_i z P_i x \;\; v \;\; z P_i y P_i x \;\; v \;\; y I_i z P_i x \;\; v \;\; y P_i z I_i x \,]$. In all cases we see that one of the four sets (A) - (D) is contained in the set of R_i.

The proof of the lemma is completed by noting that all the four sets (A) - (D) violate all three restrictions.

Theorem 1 : For every non-minority rule, a necessary and sufficient condition for quasi-transitivity of the social preference relation is that (VR v WCP v UVR) holds over every triple of alternatives.

Proof : <u>Sufficiency</u>

Suppose quasi-transitivity is violated. Then for some x, y, z $\in$ S we must have $x\ P\ y \wedge y\ P\ z \wedge \sim (x\ P\ z)$.

$$x\ P\ y \longrightarrow N(x\ P_i\ y) > p\ N \quad (1)$$

$$y\ P\ z \longrightarrow N(y\ P_i\ z) > p\ N \quad (2)$$

$$\sim(x\ P\ z) \longrightarrow N(x\ P_i\ z) \leqslant p\ N$$

$$\longrightarrow N(z\ R_i\ x) \geqslant (1-p)\ N \quad (3)$$

$$(1) \wedge (2) \longrightarrow \exists i : x\ P_i\ y\ P_i\ z, \quad (4)$$

as $\frac{1}{2} \leqslant p < 1$.

$$(2) \wedge (3) \longrightarrow \exists i : y\ P_i\ z\ R_i\ x \quad (5)$$

$$(1) \wedge (3) \longrightarrow \exists i : z\ R_i\ x\ P_i\ y \quad (6)$$

(4), (5) and (6) imply that the set of individual orderings must contain one of the following 4 sets of orderings ,

(a) $x\ P_i\ y\ P_i\ z$

$y\ P_i\ z\ P_i\ x$

$z\ P_i\ x\ P_i\ y$

(b) $x\ P_i\ y\ P_i\ z$

$y\ P_i\ z\ P_i\ x$

$z\ I_i\ x\ P_i\ y$

(c) $x \, P_i \, y \, P_i \, z$

$y \, P_i \, z \, I_i \, x$

$z \, P_i \, x \, P_i \, y$

(d) $x \, P_i \, y \, P_i \, z$

$y \, P_i \, z \, I_i \, x$

$z \, I_i \, x \, P_i \, y$

As each one of these sets violates all 3 restrictions VR, WCP and UVR it follows that (VR v WCP v UVR) is sufficient for quasi-transitivity.

Necessity

If a set of orderings violates all 3 restrictions VR, WCP and UVR then by lemma 3 it must contain one of the four sets A, B, C and D, except for a formal interchange of alternatives. Therefore, for proving the necessity of (VR v WCP v UVR) it suffices to show that for each of the four sets there exists an assignment of individuals which results in violation of quasi-transitivity. For A, C and D choose $N_1 = pN$, $N_2 = N_3 = \frac{(1-p) \, N}{2}$. For this assignment we have $N \, (x \, P_i \, y) > pN$, $N \, (y \, P_i \, z) > pN$ and $N \, (x \, P_i \, z) = pN$. So $x \, P \, y \wedge y \, P \, z \wedge \sim (x \, P \, z)$. For B choose $N_2 = pN$, $N_1 = N_3 = \frac{(1-p) \, N}{2}$. As $N \, (y \, P_i \, z) > pN$,

$N(z\ P_i\ x) > pN$ and $N(y\ P_i\ x) = pN$, this results in $y\ P\ z \wedge z\ P\ x \wedge \sim(y\ P\ x)$.

Necessary and Sufficient Conditions for Transitivity

Lemma 4 : A set of individual orderings **violates** both CP and EVR iff it includes one of the following four 2-ordering sets, except for a formal interchange of alternatives ,

(A) $x\ P_i\ y\ P_i\ z$
$y\ P_i\ z\ P_i\ x$

(B) $x\ P_i\ y\ P_i\ z$
$y\ P_i\ z\ I_i\ x$

(C) $x\ P_i\ y\ P_i\ z$
$z\ I_i\ x\ P_i\ y$

(D) $x\ P_i\ y\ I_i\ z$
$x\ I_i\ y\ P_i\ z$

Proof : From the definition of CP it follows that a set of individual orderings violates CP iff it contains one of the following 8 sets of orderings, except for a formal interchange of alternatives,

(1)	$x P_i y P_i z$ $y P_i z P_i x$	(2)	$x P_i y P_i z$ $y P_i z I_i x$
(3)	$x P_i y P_i z$ $z I_i x P_i y$	(4)	$x P_i y I_i z$ $x I_i y P_i z$
(5)	$x P_i y P_i z$ $x P_i z P_i y$	(6)	$x P_i y P_i z$ $y P_i x P_i z$
(7)	$x P_i y I_i z$ $y P_i x I_i z$	(8)	$y I_i z P_i x$ $z I_i x P_i y$

The first four sets are the same as A, B, C and D, so it suffices to consider the remaining 4 sets.

For violating EVR for (5) we must include an ordering in which x is medium without being worst or an ordering in which y is uniquely best or an ordering in which z is uniquely best. But then the set of R_i would contain one of the sets (A) - (D).

Similarly, for (6) EVR is violated iff an ordering is included in which z is medium without being best or an ordering in which x is uniquely worst or an ordering in which y is uniquely worst. With the inclusion of the required ordering the set of R_i contains one of the sets (A) - (D). (7) would violate EVR only if an R_i in which some alternative is uniquely worst is included. With the inclusion of an ordering in which some alternative is uniquely worst, excepting the cases when $x\ P_i\ z\ P_i\ y$ or $y\ P_i\ z\ P_i\ x$ is included, EVR is violated and one of (A) - (D) is contained in the set of R_i. If $x\ P_i\ z\ P_i\ y$ or $y\ P_i\ z\ P_i\ x$ is included then EVR is violated iff an ordering is included in which x or y is medium without being worst or z is uniquely best. In each of these cases the set of R_i contains one of the sets (A) - (D). Finally, (8) would violate EVR only if an R_i in which some alternative is uniquely best is included. Again we see that with the inclusion of required ordering, excepting the cases when $x \; P_i \; z \; P_i \; y$ or $y \; P_i \; z \; P_i \; x$ is included, EVR is violated and the set of R_i contains one of the sets (A) - (D). To violate EVR when the R_i included is $x \; P_i \; z \; P_i \; y$ or $y \; P_i \; z \; P_i \; x$, one must include an R_i in which x or y is medium without being best or z is uniquely worst. In each case EVR is violated and the set of R_i contains one of the four sets (A)-(D). The proof is completed by noting that each of the four sets (A) - (D) violates both CP and EVR.

Theorem 2 : For every non-minority rule, a necessary and sufficient condition for transitivity of the social R is that (CP v EVR) holds over every triple of alternatives.

Proof : Sufficiency

Let transitivity be violated. Then for some $x, y, z \in S$ we must have $x \; R \; y \wedge y \; R \; z \wedge z \; P \; x$.

$$x \, R \, y \longrightarrow N (y \, P_i \, x) \leqslant p \, N$$

$$\longrightarrow N (x \, R_i \, y) \geqslant (1-p) \, N \qquad (1)$$

Similarly,

$$y \, R \, z \longrightarrow N (y \, R_i \, z) \geqslant (1-p) \, N \qquad (2)$$

$$z \, P \, x \longrightarrow N (z \, P_i \, x) > p \, N \qquad (3)$$

$$(1) \wedge (3) \longrightarrow \exists i : z \, P_i \, x \, R_i \, y \qquad (4)$$

$$(2) \wedge (3) \longrightarrow \exists i : y \, R_i \, z \, P_i \, x \qquad (5)$$

(4) and (5) imply that the set of individual orderings must contain one of the following four sets of orderings,

(a) $z \, P_i \, x \, P_i \, y$, $y \, P_i \, z \, P_i \, x$

(b) $z \, P_i \, x \, P_i \, y$, $y \, I_i \, z \, P_i \, x$

(c) $z \, P_i \, x \, I_i \, y$, $y \, P_i \, z \, P_i \, x$

(d) $z \, P_i \, x \, I_i \, y$, $y \, I_i \, z \, P_i \, x$

As each of these sets violates both CP and EVR it follows that (CP v EVR) is sufficient for transitivity.

<u>Necessity :</u>

Let both EVR and CP be violated. Then, by lemma 4, the set of R_i must contain one of the four sets A, B, C and D of lemma 4, except for a formal interchange of alternatives. Therefore it suffices to show that for each of the four sets there exists an assignment of individuals which results in intransitive social preferences. For each case take $N_1 = N_2$. For A and B this results in $x I y \wedge y P z \wedge x I z$, for C in $x P y \wedge y I z \wedge x I z$ and for D in $x I y \wedge y I z \wedge x P z$. This establishes the necessity of (EVR v CP) for transitivity of the social R generated by a non-minority rule.

References

1. Arrow, K. J. (1963) : Social Choice and Individual Values , Wiley, N. Y., Second edition.

2. Dummett, M., and Farquharson, R. (1961) : 'Stability in Voting', Econometrica, 29.

3. Fine, K. (1973) : 'Conditions for the existence of cycles under majority and non-minority rules,' Econometrica, 41.

4. Inada, K. (1964) : 'A note on the simple majority decision rule', Econometrica, 32.

5. Inada, K. (1969) : 'The simple majority decision rule', Econometrica, 37.

6. Pattanaik, P.K. (1968) : 'A note on democratic decisions and the existence of choice sets', Review of Economic Studies, 35.

7. Pattanaik, P.K. (1971) : Voting and Collective Choice, Cambridge University Press, Cambridge.

8. Sen, A.K., and Pattanaik, P.K. (1969) : 'Necessary and sufficient conditions for rational choice under majority decision', Journal of Economic Theory, 1.

9. Sen, A.K. (1970): Collective Choice and Social Welfare, Holden-Day, San Francisco.