Necessary and Sufficient Conditions for Quasi-Transitivity and Transitivity of Pareto-Inclusive Non-Minority Rules

Satish K. Jain
Centre for Economic Studies and Planning
Jawaharlal Nehru University
New Delhi 110067
INDIA

Abstract

It is shown that (i) for every Pareto-inclusive non-minority rule a necessary and sufficient condition for quasi-transitivity is that limited agreement or Latin Square unique value restriction holds over every triple of alternatives (ii) for every special Pareto-inclusive non-minority rule a necessary and sufficient condition for transitivity is that strong value restriction or absence of unique extremal value holds over every triple of alternatives (iii) for simple Pareto-inclusive non-minority rule a necessary and sufficient condition for transitivity is that strongly echoic preferences or strong value restriction or absence of unique extremal value is satisfied over every triple of alternatives.

Necessary and Sufficient Conditions for Quasi-Transitivity and Transitivity of Pareto-Inclusive Non-Minority Rules

Satish K. Jain

In this paper we derive necessary and sufficient conditions for quasi-transitivity and transitivity of Pareto-inclusive non-minority rules. It is shown that for every Pareto-inclusive non-minority rule, limited agreement and Latin Square unique value restriction constitute a set of necessary and sufficient conditions for the quasi-transitivity of the social preference relation. Unlike the case of quasi-transitivity, conditions for transitivity do not turn out to be identical for all Pareto-inclusive non-minority rules. Whereas for simple Pareto-inclusive non-minority rule strong value restriction, strongly echoic preferences and the absence of unique extremal value constitute a set of necessary and sufficient conditions for transitivity, for special Pareto-inclusive non-minority rules a necessary and sufficient condition for transitivity is that the condition of absence of extremal value or strong value restriction holds over every triple of alternatives.

1. Restrictions on Preferences

The set of social alternatives would be denoted by S. The cardinality n of S would be assumed to be finite and greater than 2. The set of individuals and the number of individuals are designated by L and N respectively. N is assumed to be finite and greater than 2. N ( ) would stand for the number of individuals holding the preferences specified in the pa rentheses and N_k for the number of individuals holding the k-th preference ordering. Each individual $i \in L$ is assumed to have an ordering R_i defined over S. The symmetric and asymmetric parts of R_i are denoted by I_i and P_i respectively. The social preference relation is denoted by R and its symmetric and asymmetric components by I and P respectively.

Pareto-Inclusive Non-Minority Rules:

$\forall\, x,y \in S$:

$x \; R \; y \longleftrightarrow \sim [(N(y \; P_i \; x) > pN) \vee (\forall i\text{:} \; y \; R_i \; x \wedge \exists i : y \; P_i \; x)]$, where p is a fraction such that $\frac{1}{2} \leqslant p < 1$. For $p = \frac{1}{2}$ we obtain the familiar simple Pareto-inclusive non-minority rule.

An individual is defined to be concerned with respect to a triple iff he is not indifferent over every pair of alternatives belonging to the triple; otherwise he is unconcerned. For individual i, in the triple $\{x,y,z\}$, x is best iff $(x R_i y \wedge x R_i z)$; medium iff $(y R_i x R_i z \vee z R_i x R_i y)$; worst iff $(y R_i x \wedge z R_i x)$; uniquely best iff $(x P_i y \wedge x P_i z)$; uniquely medium iff $(y P_i x P_i z \vee z P_i x P_i y)$; and uniquely worst iff $(y P_i x \wedge z P_i x)$. Now we define several restrictions which specify the permissible sets of individual orderings. All these restrictions are defined over triples of alternatives.

Limited Agreement (LA) : LA holds over $\{x,y,z\}$ iff there exist distinct $a,b \in \{x,y,z\}$ such that $\forall i \in L : a R_i b$.

Latin Square Unique Value Restriction (LSUVR): There does not exist an alternative belonging to the triple such that it is uniquely medium in some R_i, uniquely best in some R_j, uniquely worst in some R_k and $\{R_i, R_j, R_k\}$ form a Latin Square. Formally, LSUVR holds over $\{x,y,z\}$ iff

$$\sim [\exists a,b,c \in \{x,y,z\} \wedge \exists i,j,k \in L : (a P_i b P_i c \wedge b P_j c R_j a \wedge c R_k a P_k b)].$$

Strong Value Restriction (SVR): SVR is satisfied over a triple iff there exists (i) an alternative such that it is best in every R_i or (ii) an alternative such that it is worst in every R_i or (iii) an alternative such that it is uniquely medium in every concerned R_i or (iv) a pair of distinct alternatives such that every individual is indifferent between the alternatives of the pair. More formally, SVR holds over $\{x,y,z\}$ iff there exist distinct $a,b,c \in \{x,y,z\}$ such that $[\forall i : (a R_i b \wedge a R_i c) \vee \forall i : (b R_i a \wedge c R_i a) \vee \forall \text{ concerned } i : (b P_i a P_i c \vee c P_i a P_i b) \vee \forall i: a I_i b]$.

Absence of Unique Extremal Value (AUEV) : There does not exist an alternative such that it is uniquely best in some R_i or there does not exist an alternative such that it is uniquely worst in some R_i. Formally, AUEV holds over $\{x,y,z\}$ iff $\sim[\exists \text{distinct } a,b,c \in \{x,y,z\}$ and $\exists i \in L : (a P_i b \wedge a P_i c)] \vee$

$\sim[\exists \text{ distinct } a,b,c \in \{x,y,z\}$ and $\exists i \in L : (b P_i a \wedge c P_i a)]$.

Strongly Echoic Preferences (SEP) : SEP holds over $\{x,y,z\}$ iff $[\sim(\exists i : x I_i y I_i z) \wedge \exists \text{distinct } a,b,c \in \{x,y,z\} : \forall i : [a R_i c \wedge (a P_i c \longrightarrow a P_i b P_i c)]]$.

2. Quasi-Transitivity of Pareto-Inclusive Non-Minority Rules

Theorem 1 : For every Pareto-inclusive non-minority rule, a necessary and sufficient condition for quasi-transitivity is that (LA v LSUVR) holds over every triple of alternatives.

Proof:

<u>Sufficiency</u>:

Suppose quasi-transitivity is violated. Then for some $x,y,z \in S$, we must have

$$x \; P \; y \wedge y \; P \; z \; \wedge \sim(x \; P \; z)$$

$$x \; P \; y \longrightarrow [N(x \; P_i \; y) > pN] \vee [\forall i: x \; R_i \; y \wedge \exists i: x \; P_i \; y] \quad (1)$$

$$y \; P \; z \longrightarrow [N(y \; P_i \; z) > pN] \vee [\forall i: y \; R_i \; z \wedge \exists i: y \; P_i \; z] \quad (2)$$

$$\sim(x \; P \; z) \longrightarrow [N(x \; P_i \; z) \leqslant pN] \wedge \sim[\forall i: x \; R_i \; z \wedge \exists i: x \; P_i \; z]$$

$$\longrightarrow [N(x \; P_i \; z) \leqslant pN] \wedge [\exists i: z \; P_i \; x \vee \forall i: z \; R_i \; x]$$

$$\longrightarrow [\exists i: z \; P_i \; x \wedge N(x \; P_i \; z) \leqslant pN] \vee \forall i: z \; R_i \; x \quad (3)$$

$$(1) \longrightarrow \exists i : x \; P_i \; y \quad (4)$$

$$(2) \longrightarrow \exists i : y \; P_i \; z \quad (5)$$

$$(3) \longrightarrow N(z \; R_i \; x) \geqslant (1-p) \; N \quad (6)$$

Suppose $\forall i: x \; R_i \; y$. By (5) then there exists an individual for whom $x \; P_i \; z$ holds. Then (3) implies that

$\exists i : z P_i x$. This in turn implies that $\exists i : z P_i y$ and therefore $N (y P_i z) > pN$. (6) together with $N (y P_i z) > pN$ implies that $\exists i : y P_i z R_i x$, i.e., there exists an individual for whom $y P_i x$ holds. This is a contradiction as we had assumed that $\forall i : x R_i y$. Therefore we conclude that $\sim (\forall i : x R_i y)$, which implies

$$\exists i : y P_i x \tag{7}$$

$$\text{and} \quad N (x P_i y) > pN \tag{8}$$

By a similar argument it can be established that $\sim (\forall i: y R_i z)$ which entails

$$\exists i : z P_i y \tag{9}$$

$$\text{and} \quad N (y P_i z) > pN \tag{10}$$

As $\frac{1}{2} \leqslant p < 1$, (8) and (10) imply that

$$\exists i : x P_i y P_i z \tag{11}$$

$$\text{(3) and (11)} \longrightarrow \exists i : z P_i x \tag{12}$$

$$\text{(6) and (8)} \longrightarrow \exists i : z R_i x P_i y \tag{13}$$

$$\text{(6) and (10)} \longrightarrow \exists i : y P_i z R_i x \tag{14}$$

(11), (13) and (14) imply that LSUVR is violated. (4), (7), (5), (9), (11) and (12) imply that LA is violated. Thus we have shown that violation of quasi-transitivity implies violation of both LSUVR and LA. Therefore we conclude that (LA v LSUVR) is sufficient for quasi-transitivity of every Pareto-inclusive non-minority rule.

Necessity:

It can be easily checked that a set of individual orderings violates both LSUVR and LA iff the set of R_i contains one of the following six sets of orderings, except for a formal interchange of alternatives. Therefore, to prove the necessity of (LA v LSUVR) for quasi-transitivity it suffices to show that for each of these sets there exists an assignment of individuals such that the social preference relation violates quasi-transitivity.

(A)
$x P_i y P_i z$
$y P_i z P_i x$
$z P_i x P_i y$

(B)
$x P_i y P_i z$
$y P_i z P_i x$
$z I_i x P_i y$

(C)
$x P_i y P_i z$
$y P_i z I_i x$
$z P_i x P_i y$

(D)
$x P_i y P_i z$
$y P_i z I_i x$
$z I_i x P_i y$
$z P_i y P_i x$

(E)
$x P_i y P_i z$
$y P_i z I_i x$
$z I_i x P_i y$
$z P_i y I_i x$

(F)
$x P_i y P_i z$
$y P_i z I_i x$
$z I_i x P_i y$
$z I_i y P_i x$

For (A), (B) and (C) take $N_1 = pN$, $N_2 = N_3 = \frac{(1-p)N}{2}$ and for (D), (E) and (F) $N_1 = pN$, $N_2 = N_3 = N_4 = \frac{(1-p)N}{3}$.
This results in $x\ P\ y \wedge y\ P\ z \wedge \sim (x\ P\ z)$.

3. Transitivity of Pareto-Inclusive Special Non-Minority Rules

Lemma 1 : A set of R_i violates SVR and AUEV iff it contains one of the following 8 sets of orderings, except for a formal interchange of alternatives,

(A) $x\ P_i\ y\ P_i\ z$
$y\ P_i\ z\ P_i\ x$

(B) $x\ P_i\ y\ P_i\ z$
$z\ P_i\ x\ I_i\ y$

(C) $x\ P_i\ y\ P_i\ z$
$y\ I_i\ z\ P_i\ x$

(D) $x\ P_i\ y\ P_i\ z$
$z\ P_i\ y\ P_i\ x$
$y\ P_i\ x\ I_i\ z$

(E) $x\ P_i\ y\ P_i\ z$
$z\ P_i\ y\ P_i\ x$
$x\ I_i\ z\ P_i\ y$

(F) $y\ P_i\ x\ I_i\ z$
$x\ I_i\ z\ P_i\ y$
$x\ P_i\ y\ P_i\ z$

(G) $y\ P_i\ x\ I_i\ z$
$x\ I_i\ z\ P_i\ y$
$x\ P_i\ y\ I_i\ z$

(H) $y\ P_i\ x\ I_i\ z$
$x\ I_i\ z\ P_i\ y$
$x\ I_i\ y\ P_i\ z$

Proof: From the definition of AUEV it follows that it is violated iff the set of R_i contains one of the following 3 sets of orderings, except for a formal interchange of alternatives,

(i) $x\ P_i\ y\ P_i\ z$ (ii) $x\ P_i\ y\ I_i\ z$ (iii) $x\ P_i\ y\ I_i\ z$

$x\ I_i\ y\ P_i\ z$ $y\ I_i\ z\ P_i\ x$

First consider (i). SVR would be violated only if an ordering in which x is not best is included. In all cases, excepting those when $y\ P_i\ x\ P_i\ z$ or $y\ P_i\ x\ I_i\ z$ or $z\ P_i\ y\ P_i\ x$ is included, SVR is violated and one of the 8 sets is included in the set of R_i. In the cases of inclusion of $y\ P_i\ x\ P_i\ z$ or $y\ P_i\ x\ I_i\ z$, SVR is violated iff an ordering in which z is not worst is included. With the inclusion of an ordering in which z is not worst, the set of R_i contains one of the 8 sets. If $z\ P_i\ y\ P_i\ x$ is included then SVR is violated iff a concerned ordering in which y is not uniquely medium is included. With the inclusion of such an ordering the set of R_i contains one of the 8 sets.

Next we consider (ii). SVR would be violated only if an ordering in which x is not best is included. In all cases excepting those when $y\ P_i\ x\ P_i\ z$ or $y\ P_i\ x\ I_i\ z$

is included, SVR is violated and the set of R_i contains one of the 8 sets. In case we include $y\ P_i\ x\ P_i\ z$ or $y\ P_i\ x\ I_i\ z$, SVR is violated iff an ordering in which z is not worst is included. With the inclusion of such an ordering the set of R_i contains one of the 8 sets.

Finally, (iii) would violate SVR iff an ordering is included in which $y\ I_i\ z$ does not hold. With the inclusion of required ordering the set of R_i contains one of the 8 sets. The proof is completed by noting that all the 8 sets violate both the restrictions.

Theorem 2 : For every Pareto-inclusive special non-minority rule ($\frac{1}{2} < p < 1$), a necessary and sufficient condition for transitivity is that (SVR v AUEV) holds over every triple of alternatives.

Proof:

<u>Sufficiency</u>:

Suppose transitivity is violated. Then for some $x,y,z \in S$ we must have $x\ R\ y \wedge y\ R\ z \wedge z\ P\ x$.

$$x\ R\ y \longrightarrow \sim y\ P\ x$$
$$\longrightarrow \sim [[N\ (y\ P_i\ x) > pN]\ v\ [\forall i:\ y\ R_i\ x \wedge \exists i:\ y\ P_i\ x]]$$

$$\longrightarrow [N\ (y\ P_i\ x) \leqslant pN] \wedge [\exists i:\ x\ P_i\ y \vee \forall i:\ x\ R_i\ y]$$

$$\longrightarrow [\exists i:\ x\ P_i\ y \wedge N(y\ P_i\ x) \leqslant pN] \vee \forall i:\ x\ R_i\ y \qquad (1)$$

Similarly, $y\ R\ z \longrightarrow [\exists i:\ y\ P_i\ z \wedge N(z\ P_i\ y) \leqslant pN] \vee$

$$\forall i:\ y\ R_i\ z \qquad (2)$$

$$z\ P\ x \longrightarrow [N\ (z\ P_i\ x) > pN] \vee [\forall i:\ z\ R_i\ x \wedge \exists i: z\ P_i\ x] \qquad (3)$$

$$(1) \longrightarrow N\ (x\ R_i\ y) \geqslant (1-p)\ N \qquad (4)$$

$$(2) \longrightarrow N\ (y\ R_i\ z) \geqslant (1-p)\ N \qquad (5)$$

$$(3) \longrightarrow \exists i:\ z\ P_i\ x \qquad (6)$$

Suppose $\forall i:\ x\ R_i\ y$. Then (6) implies that $\exists i:$ $z\ P_i\ y$ which in turn implies, by (2), that there exists an individual for whom $y\ P_i\ z$ holds. From $\forall i:\ x\ R_i\ y$ and $\exists i:\ y\ P_i\ z$ we conclude that $\exists i:\ x\ P_i\ z$ and therefore by (3), $N\ (z\ P_i\ x) > pN$. But (5) together with $N\ (z\ P_i\ x) > pN$ implies that $\exists i:\ y\ R_i\ z\ P_i\ x$ entailing that there is an individual for whom $y\ P_i\ x$ obtains, which contradicts our assumption that $\forall i:\ x\ R_i\ y$. Therefore, $(\forall i:\ x\ R_i\ y)$ is false, i.e.,

$$\exists i:\ y\ P_i\ x \qquad (7)$$

By a similar argument it can be shown that $\sim(\forall i: y\ R_i\ z)$ holds and so

$$\exists i : z\ P_i\ y \tag{8}$$

$$(7) \text{ and } (1) \longrightarrow \exists i : x\ P_i\ y \tag{9}$$

$$(8) \text{ and } (2) \longrightarrow \exists i : y\ P_i\ z \tag{10}$$

First suppose that $N\ (z\ P_i\ x) > pN$. Then by (4) and (5) we must have

$\exists i : z\ P_i\ x\ R_i\ y$ and $\exists i : y\ R_i\ z\ P_i\ x$.

This coupled with (9) and (10) implies that both AUEV and SVR are violated.

Next we assume that $(\forall i: z\ R_i\ x \wedge \exists i: z\ P_i\ x)$. Then by (9) and (10) we conclude that

$\exists i : z\ R_i\ x\ P_i\ y$ and $\exists i : y\ P_i\ z\ R_i\ x$.

This together with (6) implies that both AUEV and SVR are violated.

Thus we have shown that whenever a Pareto-inclusive special non-minority rule violates transitivity both SVR and AUEV are violated, i.e., (SVR v AUEV) is a sufficient condition for transitivity.

Necessity:

By lemma 1 a set of R_i violates AUEV and SVR iff it contains one of the 8 sets (A) - (H), except for a formal interchange of alternatives. Therefore it suffices to show that for each of these 8 sets there exists an assignment of individuals which results in intransitive social preference relation. For (A), (B) and (C) take $N_1 = N_2$, for (D), (E) and (G), $N_1 = pN$, $N_2 = (1-p)\ N-1$, $N_3 = 1$, for (F), $N_1 = pN-1$, $N_2 = (1-p)\ N$, $N_3 = 1$, where $N \geqslant \frac{1}{2p-1}$, and for (H), $N_1 = (1-p)N-1$, $N_2 = pN$, $N_3 = 1$. Then the social preference relation is, for (A), (C) and (D), $x\ I\ y \wedge y\ P\ z \wedge x\ I\ z$, for (B) and (E), $x\ P\ y \wedge y\ I\ z \wedge x\ I\ z$, and for (F), (G) and (H), $x\ I\ y \wedge y\ I\ z \wedge x\ P\ z$. Thus in each case transitivity is violated which establishes the necessity of (SVR v AUEV).

4. Transitivity of Pareto-Inclusive Simple Non-Minority Rule

Lemma 2 : A set of R_i violates all three restrictions SVR, AUEV and SEP iff it contains one of the following 8 sets of orderings, except for a formal interchange of alternatives;

(i) $x P_i y P_i z$
$y P_i z P_i x$

(ii) $x P_i y P_i z$
$z P_i x I_i y$

(iii) $x P_i y P_i z$
$y I_i z P_i x$

(iv) $x P_i y P_i z$
$z P_i y P_i x$
$y P_i x I_i z$

(v) $x P_i y P_i z$
$z P_i y P_i x$
$x I_i z P_i y$

(vi) $y P_i x I_i z$
$x I_i z P_i y$
$x P_i y I_i z$

(vii) $y P_i x I_i z$
$x I_i z P_i y$
$x I_i y P_i z$

(viii) $y P_i x I_i z$
$x I_i z P_i y$
$x P_i y P_i z$
$x I_i y I_i z$

Proof: From lemma 1 we know that a set of R_i violates SVR and AUEV iff it contains one of the 8 sets (A) - (H). Except (F), all other sets violate SEP also. Sets (i) to (vii) are the same as these sets. F would violate SEP iff an ordering not already in the set is included. With the inclusion of required ordering the set of R_i contains one of the 8 sets (i) - (viii). As (i) - (viii) violate all three restrictions, lemma is established.

Theorem 3: For Pareto-inclusive simple non-minority rule, a necessary and sufficient condition for transitivity is that (SVR v AUEV v SEP) holds over every triple of alternatives.

Proof: Suppose transitivity is violated. Then for some $x, y, z \in S$ we must have $x \; R \; y \wedge y \; R \; z \wedge z \; P \; x$.

$$x \; R \; y \longrightarrow [\exists i: x \; P_i \; y \wedge N \, (y \; P_i \; x) \leqslant N/2] \vee \forall i : x \; R_i \; y \quad (1)$$

$$y \; R \; z \longrightarrow [\exists i: y \; P_i \; z \wedge N \, (z \; P_i \; y) \leqslant N/2] \vee \forall i : y \; R_i \; z \quad (2)$$

$$z \; P \; x \longrightarrow [N \, (z \; P_i \; x) > N/2] \vee [\forall i: z \; R_i \; x \wedge \exists i : z \; P_i \; x] \quad (3)$$

$$(1) \longrightarrow N \, (x \; R_i \; y) \geqslant N/2 \quad (4)$$

$$(2) \longrightarrow N \, (y \; R_i \; z) \geqslant N/2 \quad (5)$$

$$(3) \longrightarrow \exists i: z \; P_i \; x \quad (6)$$

$$(4) \text{ and } (5) \longrightarrow [\exists i : x \; R_i \; y \; R_i \; z] \vee [N \, (x \; R_i \; y \wedge z \; P_i \; y) = N/2 \wedge N \, (y \; R_i \; z \wedge y \; P_i \; x) = N/2] \quad (7)$$

Suppose $\forall i : x \; R_i \; y$. Then (6) implies that $\exists i : z \; P_i \; y$ which implies that $\exists i : y \; P_i \; z$, in view of (2).

$\forall i: x R_i y$ and $\exists i: y P_i z$ imply that $\exists i: x P_i z$ and therefore by (3), $N(z P_i x) > N/2$. But then (5) coupled with $N(z P_i x) > N/2$ implies that $\exists i: y P_i x$ which contradicts $\forall i: x R_i y$. Thus $\forall i: x R_i y$ is impossible. By a similar reasoning $\forall i: y R_i z$ is impossible. Thus,

$$\exists i: y P_i x \quad \text{and} \tag{8}$$

$$\exists i: z P_i y \tag{9}$$

$$(1) \text{ and } (8) \longrightarrow \exists i: x P_i y \tag{10}$$

$$(2) \text{ and } (9) \longrightarrow \exists i: y P_i z \tag{11}$$

By (3) either $N(z P_i x) > N/2$ or $(\forall i: z R_i x \wedge \exists i: z P_i x)$. First suppose that $N(z P_i x) > N/2$.

$$N(z P_i x) > N/2 \wedge (4) \longrightarrow \exists i: z P_i x R_i y \tag{12}$$

$$N(z P_i x) > N/2 \wedge (5) \longrightarrow \exists i: y R_i z P_i x \tag{13}$$

(12) and (13) imply a violation of AUEV. (10) and (11) together with (12) and (13) imply that SVR and SEP are violated. Thus, $N(z P_i x) > N/2$ implies that all three restrictions are violated.

Next suppose that $\forall i: z R_i x$ and $\exists i: z P_i x$.

$$(10) \wedge \forall i: z R_i x \longrightarrow \exists i: z R_i x P_i y \tag{14}$$

$(11) \wedge \forall i: z\ R_i\ x \longrightarrow \exists i: y\ P_i\ z\ R_i\ x$ (15)

(14) and (15) imply that AUEV is violated. From (6), (14) and (15) we conclude that SVR is violated and from (6), (7), (14) and (15) that SEP is violated. Thus, $(\forall i: z\ R_i\ x \wedge \exists i: z\ P_i\ x)$ also implies that all three restrictions are violated. Therefore, (AUEV v SVR v SEP) is sufficient for transitivity.

Necessity:

By lemma 2 a set of R_i violates all three restrictions AUEV, SVR and SEP iff it includes one of the sets (i) - (viii) of lemma 2, except for a formal interchange of alternatives. We prove the necessity of (AUEV v SVR v SEP) for transitivity by showing that for each of the 8 sets there exists an assignment of individuals which results in intransitive social preferences.

For (i), (ii) and (iii) take $N_1 = N_2$, for (iv), (v) and (vi) $N_1 = N/2$, $N_2 = N/2 - 1$, $N_3 = 1$, for (vii) $N_1 = N/2 - 1$, $N_2 = N/2$, $N_3 = 1$ and for (viii) $N_1 = N_2 = N_3 = N_4 = N/4$. This results, for (i), (iii) and (iv) in $x\ I\ y \wedge y\ P\ z \wedge x\ I\ z$, for (ii) and (v) in $x\ P\ y \wedge y\ I\ z \wedge x\ I\ z$ and for (vi), (vii) and (viii) in $x\ I\ y \wedge y\ I\ z \wedge x\ P\ z$.

References

1. Arrow, K.J., Social Choice and Individual Values, Wiley, N.Y., 1951, 2nd ed. 1963.

2. Fine, K., " Conditions for the existence of cycles under majority and non-minority rules", Econometrica, 41, 1973.

3. Inada, K., ''A note on the simple majority decision rule'', Econometrica, 32, 1964.

4. Inada, K., "The simple majority decision rule", Econometrica, 37, 1969.

5. Pattanaik, P.K., ''A note on democratic decisions and the existence of choice sets'', Review of Economic Studies, 35, 1968.

6. Pattanaik, P.K., Voting and Collective Choice, Cambridge University Press, Cambridge, 1971.

7. Sen, A.K., and P.K. Pattanaik, " Necessary and sufficient conditions for rational choice under majority decision" , Journal of Economic Theory, 1, 1969.

8. Sen, A.K., Collective Choice and Social Welfare, Holden-Day, San Francisco, 1970.