Structure of Neutral and Monotonic Binary Social Decision Rules with Quasi-Transitive Individual Preferences

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Abstract

The paper investigates the structure of neutral and monotonic binary social decision rules (SDRs) with unrestricted domain under the assumption that individual weak preference relations are reflexive, connected and quasi-transitive. Among others, the following characterization theorems have been proved in the paper : (1) A binary SDR is neutral and monotonic iff it satisfies weak Pareto quasi-transitivity. (2) A neutral and monotonic binary SDR yields transitive social weak preference relation for every profile iff it is null. (3) A neutral and monotonic binary SDR yields quasi-transitive social weak preference relation for every profile iff it is null or oligarchic simple game. (4) A condition on the intersection of decisive sets is shown to be necessary and sufficient for a neutral and monotonic binary SDR to yield acyclic social weak preference relation for every profile.

Key Words : Binary Social Decision Rules, Neutrality, Monotonicity, Quasi-Transitivity, Acyclicity

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The purpose of this paper is to investigate the structure of neutral and monotonic binary social decision rules with unrestricted domain under the assumption that individual weak preference relations are reflexive, connected and quasi-transitive. The conditions of neutrality and monotonicity are perhaps two of the most fundamental requirements of democratic decision-making and have been extensively discussed in the literature in the context of Arrowian framework [See Arrow (1963), Blair and Pollak (1982), Blau (1957, 1972, 1976), Blau and Deb (1977), Gibbard (1969), Guha (1972), Hansson (1969) and Sen (1970) among others]. In these contributions it has been assumed that individual weak preference relations are reflexive, connected and transitive, i.e., are orderings. In this paper, however, we assume individual weak preference relations to be reflexive, connected and quasi-transitive and investigate the structure of neutral and monotonic binary social decision rules under this weaker assumption on individual weak preference relations. This investigation is of interest in so far as there are reasons to believe that individual weak preference relations are likely to be quasi-transitive rather than transitive [see Armstrong (1951) and Pattanaik (1971) among others].

We show that a necessary and sufficient condition for a binary social decision rule to be neutral and monotonic is that it satisfies the condition of weak Pareto quasi-transitivity. Weak Pareto quasi-transitivity is a hybrid condition similar to Pareto transitivity [Wilson (1972)], though much weaker. Under the assumption that individual weak preference relations are orderings, a necessary and sufficient condition for a Paretian binary social decision rule to be neutral and monotonic is that it satisfies the condition of Pareto quasi-transitivity [Jain (1988)]. As weak Pareto quasi-transitivity is a weaker requirement than Pareto quasi-transitivity, it follows that when the domain is enlarged to include all logically possible configurations of individual reflexive, connected and quasi-transitive weak preference relations the important properties of neutrality and monotonicity are characterized by a weaker condition than in the case when the domain consists of all logically possible configurations of individual orderings.

Under the assumption that individual weak preference relations are reflexive, connected and quasi-transitive, for the class of neutral and monotonic binary social decision rules with unrestricted domain, the following characterization theorems have been proved in the paper :

(i) A neutral and monotonic binary social decision rule yields transitive social weak preference relation for every profile of individual weak preference relations iff it is null.

(ii) A neutral and monotonic binary social decision rule yields quasi-transitive social weak preference relation for every profile of individual weak preference relations iff it is null or oligarchic simple game.

(iii) A neutral and monotonic binary social decision rule yields acyclic social weak preference relation for every profile of individual weak preference relations iff there does not exist a non-empty collection $\{V_1,V_2,...,V_m\}$ of non-empty subsets of the set of individuals N such that (a) $V_j \subseteq (N - A_j)$ is a decisive set relative to $(N - A_j)$ for some $A_j \subset N$, j = 1,...,m; [$V \subseteq (N - A)$ is defined to be a decisive set relative to $(N - A)$, $A \subset N$, iff for all pairs of distinct alternatives x,y, whenever all individuals in A are indifferent between x and y and all individuals in V prefer x to y, x is socially preferred to y];[1] (b) for each $j \in \{1,2,...,m\}$: $V_1 \cap V_2 \cap ... \cap V_{j-1} \cap (V_j \cup A_j) \cap V_{j+1} \cap ... \cap V_m = \emptyset$; (c) $3 \leq m \leq \#S$, where S is the set of social alternatives.

It is of some interest to note that none of these characterizations is valid if the domain consists only of all logically possible configurations of individual orderings.

1. Notation and Definitions

We denote the set of social alternatives by S and assume that it contains at least 3 elements. We denote the finite set of individuals by N and assume that $\#N = l \geq 2$. Each individual $i \in N$ will be assumed to have a binary weak preference relation R_i over S. The asymmetric parts of binary relations R_i, R'_i, R, R′ etc., will be denoted by P_i, P'_i, P, P′ etc., respectively; and symmetric parts by I_i, I'_i, I, I′ etc., respectively.

We define a binary relation R over a set S to be (i) reflexive iff $(\forall x \in S)$ (xRx), (ii) connected iff $(\forall x,y \in S)$ $[x \neq y \rightarrow xRy \vee yRx]$, (iii) acyclic iff $(\forall x_1,x_2,x_3,....,x_m \in S)$ $[x_1Px_2 \wedge ... \wedge x_{m-1}Px_m \rightarrow x_1Rx_m]$, where m is a positive integer ≥ 3, (iv) quasi-transitive iff $(\forall x,y,z \in S)$ $[xPy \wedge yPz \rightarrow xPz]$, (v) transitive iff $(\forall x,y,z \in S)$ $[xRy \wedge yRz \rightarrow xRz]$, (vi) an ordering iff R is reflexive, connected and transitive.

We denote by C the set of all reflexive and connected binary relations over S, by Q the set of all reflexive, connected and quasi-transitive binary relations over S, and by T the set of all orderings over S.

A social decision rule (SDR) is a function from $D \subseteq C^l$ to C; $f : D \mapsto C$. In this paper we will consider the case of $D = Q^l$. In other words, the domain of the SDR will be taken to be the set of all logically possible l-tuples $(R_1,...,R_l)$ of reflexive, connected and quasi-transitive individual binary weak preference relations. $(R_1,...,R_l)$, $(R'_1,...,R'_l)$ etc., will be written as $<R_i>$, $<R'_i>$ etc., respectively in abbreviated form. The social binary weak preference relations corresponding to $<R_i>$, $<R'_i>$ etc., will be denoted by R, R′ etc., respectively.

An SDR satisfies (i) weak Pareto-criterion (WP) iff $(\forall <R_i> \in D)$ $(\forall x,y \in S)$ $[(\forall i \in N)(xP_iy) \rightarrow xPy]$, (ii) Pareto-preference (PP) iff $(\forall <R_i> \in D)$ $(\forall x,y \in S)$ $[(\forall i \in N)(xR_iy) \wedge (\exists i \in N)(xP_iy) \rightarrow xPy]$, (iii) Parteo-indifference (PI) iff $(\forall <R_i> \in D)$ $(\forall x,y \in S)$ $[(\forall i \in N)(xI_iy) \rightarrow xIy]$, (iv) Pareto criterion ($\overline{P}$) iff Pareto-preference and Pareto-indifference hold, (v) weak Pareto quasi-transitivity (WPQT) iff $(\forall <R_i> \in D)$ $(\forall x,y,z \in S)$ $[[xPy \wedge (\forall i \in N)(yP_iz) \rightarrow xPz] \wedge [(\forall i \in N)(xP_iy) \wedge yPz \rightarrow xPz]]$, (vi) Pareto quasi-transitivity (PQT) iff $(\forall <R_i> \in D)$ $(\forall x,y,z \in S)$ $[[xPy \wedge (\forall i \in N)(yR_iz) \wedge (\exists i \in N)(yP_iz) \rightarrow xPz] \wedge [(\forall i \in N)(xR_iy) \wedge (\exists i \in N)(xP_iy) \wedge yPz \rightarrow xPz]]$, (vii) binariness or independence of irrelevant alternatives (I) iff $(\forall <R_i>,<R'_i> \in D)$ $(\forall x,y \in S)$ $[(\forall i \in N)[(xR_iy \leftrightarrow xR'_iy) \wedge (yR_ix \leftrightarrow yR'_ix)] \rightarrow [(xRy \leftrightarrow xR'y) \wedge (yRx \leftrightarrow yR'x)]]$.

A binary SDR satisfies (i) neutrality (N) iff $(\forall <R_i>,<R'_i> \in D)$ $(\forall x,y,z,w \in S)$ $[(\forall i \in N)[(xR_iy \leftrightarrow zR'_iw) \wedge (yR_ix \leftrightarrow wR'_iz)] \rightarrow [(xRy \leftrightarrow zR'w) \wedge (yRx \leftrightarrow wR'z)]]$, (ii) monotonicity (M) iff $(\forall <R_i>,<R'_i> \in D)$ $(\forall x,y \in S)$ $[(\forall i \in N)[(xP_iy \rightarrow xP'_iy) \wedge (xI_iy \rightarrow xR'_iy)] \rightarrow [(xPy \rightarrow xP'y) \wedge (xIy \rightarrow xR'y)]]$, (iii) weak monotonicity (WM) iff $(\forall <R_i>,<R'_i> \in D)$ $(\forall x,y \in S)$ $(\forall k \in N)$ $[(\forall i \in N-\{k\})[(xR_iy \leftrightarrow xR'_iy) \wedge (yR_ix \leftrightarrow yR'_ix)] \wedge [(yP_kx \wedge xR'_ky) \vee (xI_ky \wedge xP'_ky)] \rightarrow [(xPy \rightarrow xP'y) \wedge (xIy \rightarrow xR'y)]]$.

An SDR is called (i) null iff $(\forall <R_i> \in D)$ $(\forall x,y \in S)$ (xIy), (ii) dictatorial iff $(\exists j \in N)$ $(\forall <R_i> \in D)$ $(\forall x,y \in S)$ $(xP_jy \rightarrow xPy)$, (iii) oligarchic iff $(\exists V \subseteq N)$ $(\forall <R_i> \in D)$ $(\forall x,y \in S)$ $[[(\forall i \in V)(xP_iy) \rightarrow xPy] \wedge [(\forall i \in V)(xP_iy \rightarrow xRy)]]$, (iv) strictly oligarchic iff $(\exists V \subseteq N)$ $(\forall <R_i> \in D)$ $(\forall x,y \in S)$ $[[(\forall i \in V)(xP_iy) \rightarrow xPy] \wedge [(\forall i \in V)(xR_iy \rightarrow xRy)]]$.

Let $A \subset N$, $V \subseteq N$ and $A \cap V = \emptyset$. Let $x,y \in S$ and $x \neq y$. We define the set of individuals V to be (i) almost (N − A)-decisive for (x,y) $[D_{N\text{-}A}(x,y)]$ iff $(\forall <R_i> \in D)$ $[(\forall i \in A)(xI_iy) \wedge (\forall i \in V)(xP_iy) \wedge (\forall i \in N-(A \cup V))(yP_ix) \rightarrow xPy]$, (ii) (N − A)-decisive for (x,y) $[\overline{D}_{N\text{-}A}(x,y)]$ iff $(\forall <R_i> \in D)$ $[(\forall i \in A)(xI_iy) \wedge (\forall i \in V)(xP_iy) \rightarrow xPy]$, (iii) (N − A)-decisive iff it is (N − A)-decisive for every $(a,b) \in S \times S$, $a \neq b$, (iv) almost (N − A)-semidecisive for (x,y) $[S_{N\text{-}A}(x,y)]$ iff $(\forall <R_i> \in D)$ $[(\forall i \in A)(xI_iy) \wedge (\forall i \in V)(xP_iy) \wedge (\forall i \in N-(A \cup V))(yP_ix) \rightarrow xRy]$, (v) (N − A)-semidecisive for (x,y) $[\overline{S}_{N\text{-}A}(x,y)]$ iff $(\forall <R_i> \in D)$ $[(\forall i \in A)(xI_iy) \wedge (\forall i \in V)(xP_iy) \rightarrow xRy]$, (vi) (N − A)-semidecisive iff it is (N − A)-semidecisive for every $(a,b) \in S \times S$, $a \neq b$.

If $A = \emptyset$, then we drop the prefix (N − A). (i) through (vi) then give definitions of an almost decisive set for (x,y) [D(x,y)], a decisive set for (x,y) [$\overline{D}$ (x,y)], a decisive set, an almost semidecisive set for (x,y) [S(x,y)], a semidecisive set for (x,y) [$\overline{S}$ (x,y)], and a semidecisive set respectively.

$V \subseteq N$ is defined to be a minimal decisive set iff it is a decisive set and no proper subset of it is a decisive set.

We denote by W the set of all decisive sets. An SDR is a simple game iff $(\forall <R_i> \in D)$ $(\forall x,y \in S)$ $[xPy \leftrightarrow (\exists V \in W)(\forall i \in V)(xP_iy)]$.

2. Characterization of Neutrality and Monotonicity

Lemma 1 : If a social decision rule $f : Q^l \mapsto C$ satisfies independence of irrelevant alternatives and weak Pareto quasi-transitivity, then it satisfies the condition of Pareto-indifference.
Proof : Suppose $f : Q^l \mapsto C$ satisfies condition I and WPQT but violates the condition of Pareto-indifference. Then,
$(\exists <R_i> \in Q^l)$ $(\exists x,y \in S)$ $[(\forall i \in N)(xI_iy) \wedge xPy]$.

By condition I we conclude,
$(\forall <R_i> \in Q^l) [(\forall i \in N) (xI_iy) \rightarrow xPy]$. (i)
Let z be an alternative distinct from x and y, and consider the configuration $(\forall i \in N) (xI_iy \wedge yP_iz \wedge xI_iz)$. We conclude by (i) and WPQT xPz and by Condition I,
$(\forall <R_i> \in Q^l) [(\forall i \in N) (xI_iz) \rightarrow xPz]$ (ii)
Next we consider the configuration $(\forall i \in N) (zP_ix \wedge xI_iy \wedge zI_iy)$. It follows from (i) and WPQT that zPy holds. From zPy and Condition I, we conclude,
$(\forall <R_i> \in Q^l) [(\forall i \in N) (zI_iy) \rightarrow zPy]$ (iii)
Finally consider the configuration $(\forall i \in N) (zI_iy \wedge yP_ix \wedge zI_ix)$. (iii) and WPQT imply zPx, and zPx and Condition I imply
$(\forall <R_i> \in Q^l) [(\forall i \in N) (xI_iz) \rightarrow zPx]$. (iv)
As (ii) and (iv) contradict each other, the lemma is established.

Lemma 2 : Let the social decision rule $f : Q^l \mapsto C$ satisfy independence of irrelevant alternatives and weak Pareto quasi-transitivity. Then $(\forall <R_i> \in Q^l) (\forall x,y \in S) [(\forall i \in N) (xR_iy) \wedge (\exists j \in N) (xP_jy) \rightarrow xRy]$.
Proof : Suppose the lemma is false. Then $(\exists <R_i> \in Q^l) (\exists x,y \in S) (\exists$ nonempty $N_1 \subseteq N) [(\forall i \in N_1) (xP_iy) \wedge (\forall i \in N - N_1) (xI_iy) \wedge yPx]$. Let z be an alternative distinct from x and y, and consider the following configuration of individual preferences,
$(\forall i \in N_1) [xP_iy \wedge xP_iz \wedge yI_iz]$
$(\forall i \in N - N_1) [xI_iy \wedge xP_iz \wedge yI_iz]$.
yPx and $(\forall i \in N) (xP_iz)$ imply yPz by WPQT. This, however, contradicts the result of Lemma 1 that the Pareto-indifference condition holds. This contradiction establishes the lemma.

Lemma 3 : Let the social decision rule $f : Q^l \mapsto C$ satisfy independence of irrelevant alternatives and weak Pareto quasi-transitivity. Then, whenever a group of individuals V is almost (N – A)-decisive for some ordered pair of distinct alternatives, it is (N – A)-decisive for every ordered pair of distinct alternatives, where $A \subset N$, $V \subseteq N$ and $A \cap V = \emptyset$.
Proof : Let V be almost (N – A)-decisive for (x,y), $x \neq y$, $x,y \in S$. Let z be an alternative distinct from x and y, and consider the following configuration of individual preferences :
$(\forall i \in A) [xI_iy \wedge yP_iz \wedge xI_iz]$
$(\forall i \in V) [xP_iy \wedge yP_iz \wedge xP_iz]$
$(\forall i \in N - (A \cup V)) [yP_ix \wedge yP_iz]$.
In view of the almost (N – A)-decisiveness of V for (x,y) and the fact that $[(\forall i \in A) (xI_iy) \wedge (\forall i \in V) (xP_iy) \wedge (\forall i \in N - (A \cup V)) (yP_ix)]$, we obtain xPy. From xPy and $(\forall i \in N) (yP_iz)$ we conclude xPz by WPQT. As $(\forall i \in A) (xI_iz)$, $(\forall i \in V) (xP_iz)$, and the preferences of individuals in $N - (A \cup V)$ have not been specified over {x,z}, it follows, in view of condition I, that V is (N – A)-decisive for (x,z). Similarly, by considering the configuration $[(\forall i \in A) (zP_ix \wedge xI_iy \wedge zI_iy) \wedge (\forall i \in V) (zP_ix \wedge xP_iy \wedge zP_iy) \wedge (\forall i \in N - (A \cup V)) (zP_ix \wedge yP_ix)]$, we can show $[D_{N\text{-}A}(x,y) \rightarrow \overline{D}_{N\text{-}A}(z,y)]$. By appropriate interchanges of alternatives it follows that $D_{N\text{-}A}(x,y) \rightarrow \overline{D}_{N\text{-}A}(a,b)$, for all $(a,b) \in \{x,y,z\} \times \{x,y,z\}$, where $a \neq b$. To prove the assertion for any $(a,b) \in S \times S$, $a \neq b$, first we note that if $[(a = x \vee a = y) \vee (b = x \vee b = y)]$, the desired conclusion $\overline{D}_{N\text{-}A}(a,b)$ can be obtained by considering a triple which includes all of x,y, a and b. If both a and b are different from x and y, then one first considers the triple {x,y,a} and deduces $\overline{D}_{N\text{-}A}(x,a)$ and hence $D_{N\text{-}A}(x,a)$, and then considers the triple {x,a,b} and obtains $\overline{D}_{N\text{-}A}(a,b)$.

Lemma 4 : Let social decision rule $f : Q^l \mapsto C$ satisfy independence of irrelevant alternatives and weak Pareto quasi-transitivity. Then, whenever a group of individuals V is almost (N – A)-semidecisive for some ordered pair of distinct alternatives, it is (N – A)-semidecisive for every ordered pair of distinct alternatives, where $A \subset N$, $V \subseteq N$ and $A \cap V = \emptyset$.
Proof : Let V be almost (N – A)-semidecisive for (x,y), $x \neq y$, $x,y \in S$. Let z be an alternative distinct from x and y, and consider the following configuration of individual preferences :
$(\forall i \in A) [xI_iy \wedge yP_iz \wedge xI_iz]$
$(\forall i \in V) [xP_iy \wedge yP_iz \wedge xP_iz]$
$(\forall i \in N - (A \cup V)) [yP_ix \wedge yP_iz]$.
From the almost (N – A)-semidecisiveness of V for (x,y), we obtain xRy. Suppose zPx. $(\forall i \in N) (yP_iz)$ and zPx imply yPx by WPQT, which contradicts xRy. Therefore zPx cannot be true, which by connectedness of R implies xRz. As $(\forall i \in A) (xI_iz)$, $(\forall i \in V) (xP_iz)$ and the preferences of individuals belonging to $N - (A \cup V)$ have not been specified over {x,z}, xRz implies $\overline{S}_{N\text{-}A}(x,z)$ in view of Condition I. Thus $S_{N\text{-}A}(x,y) \rightarrow \overline{S}_{N\text{-}A}(x,z)$. Similarly, by considering the configuration $[(\forall i \in A) (zP_ix \wedge xI_iy \wedge zI_iy), (\forall i \in V) (zP_ix \wedge xP_iy \wedge zP_iy),$

$(\forall i \in N-(A\cup V))$ $(zP_ix \wedge yP_ix)]$, we can show that $S_{N\text{-}A}(x,y) \rightarrow \overline{S}_{N\text{-}A}(z,y)$. By appropriate interchanges of alternatives it follows that $S_{N\text{-}A}(x,y)$ implies $\overline{S}_{N\text{-}A}(a,b)$ for all $(a,b) \in \{x,y,z\} \times \{x,y,z\}$, $a \neq b$. Now, the rest of the proof establishing that $S_{N\text{-}A}(x,y)$ implies $\overline{S}_{N\text{-}A}(a,b)$ for all $(a,b) \in S \times S$, $a \neq b$, is similar to that of lemma 3 and will be omitted here.

Proposition 1 : If a social decision rule $f : Q^l \mapsto C$ satisfies independence of irrelevant alternatives and weak Pareto quasi-transitivity then it is neutral.
Proof : Consider any $<R_i>$, $<R'_i> \in Q^l$ such that $(\forall i \in N)$ $[(xR_iy \leftrightarrow zR'_iw) \wedge (yR_ix \leftrightarrow wR'_iz)]$, $x,y,z,w \in S$. Designate by N_1, N_2 and N_3 the sets $\{i \in N \mid xP_iy \wedge zP'_iw\}$, $\{i \in N \mid xI_iy \wedge zI'_iw\}$ and $\{i \in N \mid yP_ix \wedge wP'_iz\}$ respectively.

If $N_1 \cup N_3 = \emptyset$, then xIy and $zI'w$ follow from the condition of Pareto-indifference which holds in view of lemma 1.

Now assume that $N_1 \cup N_3 \neq \emptyset$. Nonemptiness of $N_1 \cup N_3$ implies that $x \neq y$ and $z \neq w$. Suppose xPy. Then N_1 is almost $(N-N_2)$-decisive for (x,y). In view of lemma 3 it follows that N_1 is $(N-N_2)$-decisive for every ordered pair of distinct alternatives. Therefore $zP'w$ must hold as $[(\forall i \in N_2)$ $(zI'_iw) \wedge (\forall i \in N_1)$ $(zP'_iw)]$. We have shown that $(xPy \rightarrow zP'w)$. By an analogous argument it can be shown that $(zP'w \rightarrow xPy)$. So we have $(xPy \leftrightarrow zP'w)$. Next suppose that yPx. Then N_3 is an almost $(N-N_2)$-decisive set for (y,x) and hence an $(N-N_2)$-decisive set by lemma 3. Therefore, we must have $wP'z$ as $[(\forall i \in N_2)$ $(wI'_iz) \wedge (\forall i \in N_3)$ $(wP'_iz)]$. So $(yPx \rightarrow wP'z)$. By a similar argument one obtains $(wP'z \rightarrow yPx)$. Therefore we have $(yPx \leftrightarrow wP'z)$. As $(xPy \leftrightarrow zP'w)$ and $(yPx \leftrightarrow wP'z)$, by the connectedness of R and R' it follows that $(xIy \leftrightarrow zI'w)$. This establishes that the SDR is neutral.

Lemma 5 : Let the social decision rule $f : Q^l \mapsto C$ satisfy the condition of independence of irrelevant alternatives. Then f is monotonic iff it is weakly monotonic.
Proof : By the definitions of monotonicity and weak monotonicity, if f is monotonic then it is weakly monotonic. Now suppose that f is weakly monotonic. Consider any $<R_i>$, $<R'_i> \in Q^l$ and any $x,y \in S$ such that $(\forall i \in N)$ $[(xPy \rightarrow xP'_iy) \wedge (xIy \rightarrow xR'_iy)]$. Let $N' = \{j_1,j_2,....,j_m\}$ be the set of individuals for whom $R_i \neq R'_i$. Rename $<R_i>$ as $<R^0_i>$ and construct $<R^t_i>$, $t = 1,...,m$, as follows :
$(\forall i \in N-\{j_t\})$ $[R^t_i = R^{t-1}_i]$, $R^t_{j_t} = R'_{j_t}$.
So we have $<R^m_i> = <R'_i>$. By Condition I and weak monotonicity, we obtain
$[(xP^{t-1}y \rightarrow xP^ty) \wedge (xI^{t-1}y \rightarrow xR^ty)]$
for $t = 1,...,m$, which implies $[(xPy \rightarrow xP'y) \wedge (xIy \rightarrow xR'y)]$. This establishes that f is monotonic.

Proposition 2 : If a social decision rule $f : Q^l \mapsto C$ satisfies independence of irrelevant alternatives and weak Pareto quasi-transitivity then it is monotonic.
Proof : Let SDR $f : Q^l \mapsto C$ satisfy Conditions I and WPQT. In view of lemma 5 it suffices to show that the SDR is weakly monotonic. Consider any $<R_i>$, $<R'_i> \in Q^l$, any $x,y \in S$ and any individual $k \in N$ such that $(\forall i \in N-\{k\})$ $[(xR_iy \leftrightarrow xR'_iy) \wedge (yR_ix \leftrightarrow yR'_ix)]$ and $[(yP_kx \wedge xR'_ky) \vee (xI_ky \wedge xP'_ky)]$. Designate by N_1, N_2 and N_3 the sets $\{i \in N \mid xP_iy\}$, $\{i \in N \mid xI_iy\}$, $\{i \in N \mid yP_ix\}$ respectively.

If $N_1 \cup N_3 = \emptyset$ then xIy and $xR'y$ follow from lemmas 1 and 2 respectively.

Now let $N_1 \cup N_3 \neq \emptyset$.

Suppose xPy. Then N_1 is almost $(N-N_2)$-decisive for (x,y) as a consequence of Condition I, and hence an $(N-N_2)$-decisive set in view of lemma 3. If $k \in N_3$ then it follows that we must have $xP'y$. Now suppose $k \in N_2$. If $yR'x$ then N_3 is almost $(N-(N_2-\{k\}))$-semidecisive for (y,x) and hence an $(N-(N_2-\{k\}))$-semidecisive set by lemma 4. As $[(\forall i \in N_3)$ $(yP_ix) \wedge (\forall i \in N_2-\{k\})$ $(xI_iy)]$, it follows that we must have yRx. This, however, contradicts the hypothesis that xPy holds. So $yR'x$ is impossible and therefore $xP'y$ must obtain.

Next suppose xIy. Then N_1 is an $(N-N_2)$-semidecisive set in view of lemma 4. If $k \in N_3$ then it follows that we must have $xR'y$ as $[(\forall i \in N_1)$ $(xP'_iy) \wedge (\forall i \in N_2)$ $(xI'_iy)]$. Suppose $k \in N_2$ and $yP'x$. $yP'x$ implies that N_3 is an $(N-(N_2-\{k\}))$-decisive set, which in turn implies that yPx must obtain as $[(\forall i \in N_3)$ $(yP_ix) \wedge (\forall i \in N_2-\{k\})$ $(xI_iy)]$, contradicting the hypothesis of xIy. So $yP'x$ is impossible and by the connectedness of R' we conclude that $xR'y$ must hold. Thus we have shown that $[(xPy \rightarrow xP'y) \wedge (xIy \rightarrow xR'y)]$, which establishes that the SDR is weakly monotonic.

Proposition 3 : Let social decision rule $f : Q^l \mapsto C$ satisfy independence of irrelevant alternatives, neutrality and monotonicity. Then f satisfies weak Pareto quasi-transitivity.

Proof : Consider any $x,y,z \in S$ and any $<R_i> \in Q^l$ such that $[xPy \wedge (\forall i \in N)(yP_i z)]$. Designate by N_1, N_2 and N_3 the sets $\{i \in N \mid xP_i y\}$, $\{i \in N \mid xI_i y\}$, $\{i \in N \mid yP_i x\}$ respectively and by N_1', N_2' and N_3' the sets $\{i \in N \mid xP_i z\}$, $\{i \in N \mid xI_i z\}$, $\{i \in N \mid zP_i x\}$ respectively. As individual weak preference relations are quasi-transitive, from $(\forall i \in N)(yP_i z)$ we conclude that $N_1 \subseteq N_1'$ and $N_3' \subseteq N_3$. Let $<R_i'> \in Q^l$ be any configuration such that $[(\forall i \in N_1)(xP_i'z) \wedge (\forall i \in N_2)(xI_i'z) \wedge (\forall i \in N_3)(zP_i'x)]$. As xPy, we conclude $xP'z$ by conditions I and N. $xP'z$ in turn implies xPz in view of $N_1 \subseteq N_1'$ and $N_3' \subseteq N_3$, as a consequence of conditions I and M. Thus we have shown that xPy and $(\forall i \in N)(yP_i z)$ imply xPz. By an analogous argument it can be shown that $(\forall i \in N)(xP_i y)$ and yPz imply xPz. This establishes that WPQT holds.

Combining Propositions 1, 2 and 3 we obtain :

Theorem 1 : A binary social decision rule $f : Q^l \mapsto C$ is neutral and monotonic iff weak Pareto quasi-transitivity holds.

3. Characterization of Transitivity

Theorem 2 : A neutral and monotonic binary social decision rule $f : Q^l \mapsto C$ yields transitive social weak preference relation for every $<R_i> \in Q^l$ iff it is null.

Proof : If f is null then obviously social weak preference relation is transitive for every $<R_i> \in Q^l$.

Let f yield transitive social weak preference relation for every $<R_i> \in Q^l$. Suppose for some $<R_i> \in Q^l$ and some $x,y \in S$, xPy obtains. Designate by N_1, N_2 and N_3 the sets $\{i \in N \mid xP_i y\}$, $\{i \in N \mid xI_i y\}$, $\{i \in N \mid yP_i x\}$ respectively. Let $z \in S$ be an alternative distinct from x and y. Consider any $<R_i'> \in Q^l$ such that $[(\forall i \in N_1)(xP_i'y) \wedge (\forall i \in N_2)(xI_i'y) \wedge (\forall i \in N_3)(yP_i'x) \wedge (\forall i \in N)(yI_i'z \wedge xI_i'z)]$. We obtain $xP'y$ by hypothesis and Condition I, and $yI'z$ by Conditions I and N. $xP'y$ and $yI'z$ imply $xP'z$ by transitivity. But this is a contradiction as $xI'z$ holds in view of $(\forall i \in N)(xI_i'z)$, by Conditions I and N. Therefore we conclude that there do not exist any $<R_i> \in Q^l$ and $x,y \in S$ such that xPy, i.e., f is null.

In the proof of Theorem 2, monotonicity has not been used, and neutrality has been used only to infer Pareto-indifference. Therefore, the following theorem holds :

Theorem 3 : Let $f : Q^l \mapsto C$ be a binary social decision rule satisfying the condition of Pareto-indifference. Then f yields transitive social weak preference relation for every $<R_i> \in Q^l$ iff it is null.

4. Alternative Characterization of the Null Social Decision Rule[2]

Weak Pareto quasi-transitivity characterizes neutrality and monotonicity for the class of binary SDRs $f : Q^l \mapsto C$. The stronger condition of Pareto quasi-transitivity characterizes neutrality and monotonicity for the class of Paretian binary SDRs $f : T^l \mapsto C$. In the context of binary SDRs $f : Q^l \mapsto C$, however, Pareto quasi-transitivity characterizes the null social decision rule as is shown by the following theorem.

Theorem 4 : Let social decision rule $f : Q^l \mapsto C$ satisfy independence of irrelevant alternatives. Then f is null iff Pareto quasi-transitivity holds.

Proof : If f is null then $(\forall <R_i> \in Q^l)(\forall x,y \in S)[\sim (xPy)]$, and therefore PQT is trivially satisfied.

Let PQT hold. Suppose for some $<R_i> \in Q^l$ and some $x,y \in S$, xPy holds. First we show that this implies that there must exist an $<R_i'> \in Q^l$ such that $xP'y$ obtains and $\{i \in N \mid xP_i'y\} \neq N$. If $\{i \in N \mid xP_i y\} \neq N$ then there is nothing to prove. Suppose $\{i \in N \mid xP_i y\} = N$ and let $(N', N - N')$ be a partition of N such that both N' and $N - N'$ are nonempty. Let z be an alternative distinct from x and y, and consider the following configuration of individual preferences :

$(\forall i \in N')[xP_i^1 y \wedge yP_i^1 z \wedge xP_i^1 z]$

$(\forall i \in N - N')[xP_i^1 y \wedge yI_i^1 z \wedge xI_i^1 z]$.

We have xP^1y by hypothesis and Condition I, and $[(\forall i \in N)(yR_i^1 z) \wedge (\exists i \in N)(yP_i^1 z)]$ by construction. So by PQT we must have xP^1z. Next consider the following configuration :

$(\forall i \in N')[(xP_i^2 y \wedge zP_i^2 y \wedge xP_i^2 z]$

$(\forall i \in N - N')[xI_i^2 y \wedge yI_i^2 z \wedge xI_i^2 z]$.

We have xP^2z by our demonstration and Condition I, and $[(\forall i \in N)(zR_i^2y) \wedge (\exists i \in N)(zP_i^2y)]$ by construction. Therefore by PQT we must have xP^2y. This establishes the claim.

Let $\langle R_i' \rangle \in Q^l$ be any configuration such that $xP'y$ and $\{i \in N \mid xP_i'y\} \neq N$. Designate by N_1, N_2 and N_3 the sets $\{i \in N \mid xP_i'y\}$, $\{i \in N \mid xI_i'y\}$, $\{i \in N \mid yP_i'x\}$ respectively. Let z be an alternative distinct from x and y, and consider the following configuration of individual preferences :

$(\forall i \in N_1)\,[xP_i''y \wedge yI_i''z \wedge xI_i''z]$

$(\forall i \in N_2)\,[xI_i''y \wedge yP_i''z \wedge xI_i''z]$

$(\forall i \in N_3)\,[yP_i''x \wedge yP_i''z \wedge xI_i''z]$.

We have $xP''y$ by hypothesis and Condition I, and $[(\forall i \in N)(yR_i''z) \wedge (\exists i \in N)(yP_i''z)]$ in view of the fact that $N_1 \neq N$. Therefore we must have $xP''z$ by PQT. As PQT implies WPQT, it contradicts the result of lemma 1 that Pareto-indifference holds. Therefore it cannot be the case that $(\exists \langle R_i \rangle \in Q^l)(\exists x,y \in S)(xPy)$, i.e., f must be null.

5. Characterization of Quasi-Transitivity

Theorem 5 : A neutral and monotonic binary social decision rule $f : Q^l \mapsto C$ yields quasi-transitive social weak preference relation for every $\langle R_i \rangle \in Q^l$ iff it is null or an oligarchic simple game[3].

Proof : If f is null then social weak preference relation is transitive for every $\langle R_i \rangle \in Q^l$. Let f be an oligarchic simple game. Consider any $\langle R_i \rangle \in Q^l$ and any $x,y,z \in S$ such that xPy and yPz. Let V be the oligarchy. Let $V_1 = \{i \in N \mid xP_iy\}$ and $V_2 = \{i \in N \mid yP_iz\}$. Then it follows that V_1 and V_2 are decisive sets as f is a simple game. In view of the fact that f is oligarchic it follows that both V_1 and V_2 contain V. Thus $(\forall i \in V)(xP_iy \wedge yP_iz)$. As individual weak preference relations are quasi-transitive, it follows that $(\forall i \in V)(xP_iz)$. So xPz must hold. This establishes that social weak preference relation is quasi-transitive for every $\langle R_i \rangle \in Q^l$.

If f yields transitive social weak preference relation for every $\langle R_i \rangle \in Q^l$ then f is null by theorem 2. Suppose that f yields quasi-transitive social weak preference relation for every $\langle R_i \rangle \in Q^l$ but does not yield transitive social weak preference relation for every $\langle R_i \rangle \in Q^l$. Then for some $\langle R_i \rangle \in Q^l$ and some $x,y,z \in S$ we must have $(xPy \wedge yIz \wedge xIz)$. xPy implies by conditions I, M and N that the set of all individuals N is a decisive set and therefore WP holds. As a consequence of WP there exists a nonempty set $V \in W$ such that V is minimally decisive. In view of the fact that f satisfies Conditions I, M, N and WP, and yields quasi-transitive social weak preference relation for every $\langle R_i \rangle \in Q^l$, it follows, by an argument analogous to the one in the proof of Gibbard's theorem [Gibbard (1969)], that V is the unique minimal decisive set.

Let $j \in V$. We will show that $(\forall \langle R_i \rangle \in Q^l)(\forall x,y \in S)[xR_jy \rightarrow xRy]$. Suppose not. Then $(\exists \langle R_i \rangle \in Q^l)(\exists x,y \in S)[xR_jy \wedge yPx]$. Then by Conditions I, M and N we conclude that :

$(\forall \langle R_i \rangle \in Q^l)(\forall x,y \in S)[xI_jy \wedge (\forall i \in N-\{j\})(yP_ix) \rightarrow yPx]$. (i)

Now consider the following configuration of individual preferences :

$[(\forall i \in N-\{j\})(zP_iy \wedge yP_ix \wedge zP_ix) \wedge (zI_jy \wedge yI_jx \wedge xP_jz)]$,

where $x,y,z \in S$ are all distinct. We obtain zPy and yPx by (i), which in turn imply zPx by quasi-transitivity. zPx implies that $N-\{j\}$ is a decisive set, by Conditions I, M and N. Therefore there exists a nonempty $V' \subseteq N-\{j\}$ such that V' is minimally decisive. As $V' \neq V$, it contradicts the fact that V is the unique minimal decisive set. Therefore we conclude that it is impossible that for some $\langle R_i \rangle \in Q^l$ and some $x,y \in S$, xR_jy and yPx hold, i.e., V is a strict oligarchy.

Consider any $\langle R_i \rangle \in Q^l$ and any $x,y \in S$. Suppose xPy holds. As yR_ix for some $i \in V$ would imply yRx as shown above, we conclude that $(\forall i \in V)(xP_iy)$. Consequently $\{i \in N \mid xP_iy\}$ is a decisive set. Thus we have shown that

$(\forall \langle R_i \rangle \in Q^l)(\forall x,y \in S)[xPy \leftrightarrow (\exists V' \in W)(\forall i \in V')(xP_iy)]$,

which proves that f is a simple game.

The conjunction of WP and quasi-transitivity implies WPQT. WPQT in turn implies neutrality and monotonicity. As in the proof of the first part of theorem 5, neutrality and monotonicity have not been used, we conclude that the following theorem holds :

Theorem 6 : Let $f : Q^l \mapsto C$ be a binary social decision rule satisfying the weak Pareto-criterion. Then, f yields quasi-transitive social weak preference relation for every $\langle R_i \rangle \in Q^l$ iff f is an oligarchic simple game.

Remark 1 : An SDR $f : Q^l \mapsto C$ is an oligarchic simple game iff it is strictly oligarchic.
Proof : Suppose $f : Q^l \mapsto C$ is an oligarchic simple game. Let V be the oligarchy. Then V is the unique minimal decisive set. Consequently $(\forall V' \in W)\ (V \subseteq V')$. As f is a simple game it follows that $(\forall \langle R_i \rangle \in Q^l)\ (\forall x,y \in S)\ [xPy \leftrightarrow (\exists V' \in W)\ (\forall i \in V')\ (xP_iy)]$. This implies that $(\forall \langle R_i \rangle \in Q^l)\ (\forall x,y \in S)\ [xPy \leftrightarrow (\forall i \in V)\ (xP_iy)]$, as $V \in W$ and $(\forall V' \in W)\ (V \subseteq V')$. This in turn implies $(\forall \langle R_i \rangle \in Q^l)\ (\forall x,y \in S)\ [yRx \leftrightarrow (\exists i \in V)\ (yR_ix)]$, which establishes that f is strictly oligarchic.

Now suppose that f is strictly oligarchic. Then f is obviously oligarchic. Let V be the strict oligarchy. Then by the definition of strict oligarchy, we obtain $(\forall \langle R_i \rangle \in Q^l)\ (\forall x,y \in S)\ [(\exists i \in V)\ (yR_ix) \rightarrow yRx]$, which is equivalent to $(\forall \langle R_i \rangle \in Q^l)\ (\forall x,y \in S)\ [xPy \rightarrow (\forall i \in V)\ (xP_iy)]$. As $V \in W$ it follows that $(\forall \langle R_i \rangle \in Q^l)\ (\forall x,y \in S)\ [xPy \rightarrow (\exists V' \in W)\ (\forall i \in V')\ (xP_iy)]$. From the definition of a decisive set then it follows that $(\forall \langle R_i \rangle \in Q^l)\ (\forall x,y \in S)\ [xPy \leftrightarrow (\exists V' \in W)\ (\forall i \in V')\ (xP_iy)]$. This establishes that f is a simple game.

In view of the above remark, the expression `oligarchic simple game' in the statements of theorems 5 and 6 can be replaced by the expression `strict oligarchy'.

6. Characterization of Acyclicity[4]

Theorem 7 : Let $f : Q^l \mapsto C$ be a neutral and monotonic binary social decision rule. Then, f yields acyclic social weak preference relation for every $\langle R_i \rangle \in Q^l$ iff there does not exist a nonempty collection $\{V_1,...,V_m\}$ of nonempty subsets of the set of individuals N such that :
(a) for each $j \in \{1,2,...,m\}$, V_j is $(N - A_j)$-decisive for some $A_j \subset N$, $V_j \cap A_j = \emptyset$,
(b) for each $j \in \{1,2,....,m\}$, $(V_j \cup A_j) \cap V_{\sim j} = \emptyset$; where $V_{\sim j}$, j =1,2,...,m, is defined by : $V_{\sim j} = \bigcap_k V_k$, $k \in \{1,2,...,m\} - \{j\}$,
(c) $3 \leq m \leq \#S$.
Proof : Suppose acyclicity is violated. Then for some $\langle R_i \rangle \in Q^l$ and some distinct $x_1,x_2,...,x_m \in S$ we must have $(x_1Px_2 \wedge ... \wedge x_{m-1}Px_m \wedge x_mPx_1)$, where $3 \leq m \leq \#S$. Let $A_j = \{i \in N \mid x_jI_ix_{j+1}\}$, j = 1,2,...,(m-1); $A_m = \{i \in N \mid x_mI_ix_1\}$; $V_j = \{i \in N \mid x_jP_ix_{j+1}\}$, j = 1,2,...,(m-1); and $V_m = \{i \in N \mid x_mP_ix_1\}$. Thus we have $V_j \cap A_j = \emptyset$, j = 1,2,...,m. By neutrality and monotonicity it follows that for each $j \in \{1,2,...,m\}$, V_j is nonempty and consequently $A_j \neq N$. By a further appeal to monotonicity and neutrality we conclude that V_j is $(N - A_j)$-decisive, j =1,2,...,m. As individual weak preference relations are quasi-transitive, it follows that $(\forall i \in V_{\sim j})\ (x_{j+1}P_ix_j)$, j = 1,2,...,(m-1), and $(\forall i \in V_{\sim m})\ (x_1P_ix_m)$. As $[(\forall i \in A_j)\ (x_jI_ix_{j+1}) \wedge (\forall i \in V_j)\ (x_jP_ix_{j+1})]$, j = 1,2,...,(m-1); and $[(\forall i \in A_m)\ (x_mI_ix_1) \wedge (\forall i \in V_m)\ (x_mP_ix_1)]$, it follows that $(A_j \cup V_j) \cap V_{\sim j} = \emptyset$, j =1,2,...,m. This proves that the violation of acyclicity implies the existence of a nonempty collection $\{V_1,...,V_m\}$ of nonempty subsets of the set of individuals N satisfying (a), (b) and (c) mentioned in the statement of the theorem.

Next suppose that there exists a nonempty collection $\{V_1,...,V_m\}$ of nonempty subsets of N such that (a), (b) and (c) hold. Consider the following configuration of individual preferences :
$(\forall i \in A_1)\ (x_1I_ix_2) \wedge (\forall i \in V_1)\ (x_1P_ix_2)$
$(\forall i \in A_2)\ (x_2I_ix_3) \wedge (\forall i \in V_2)\ (x_2P_ix_3)$
......
......
$(\forall i \in A_{m-1})\ (x_{m-1}I_ix_m) \wedge (\forall i \in V_{m-1})\ (x_{m-1}P_ix_m)$
$(\forall i \in A_m)\ (x_mI_ix_1) \wedge (\forall i \in V_m)\ (x_mP_ix_1)$.
It is possible to have the above configuration of preferences without violating quasi-transitivity of individual weak preference relations because for each $j \in \{1,2,...,m\}$, it is given that $(V_j \cup A_j) \cap V_{\sim j} = \emptyset$. As V_j is $(N - A_j)$-decisive, j = 1,2,...,m, we conclude that $(x_1Px_2 \wedge x_2Px_3 \wedge ... \wedge x_{m-1}Px_m \wedge x_mPx_1)$ holds, which violates acyclicity. This establishes the theorem.

Footnotes

1. Throughout this paper we use the following notation :
For any sets A and B,
$A \subseteq B$ iff $(\forall x)$ $(x \in A \rightarrow x \in B)$
$A \subset B$ iff $A \subseteq B \wedge A \neq B$.

2. For characterization of null social decision rule when the domain consists of all logically possible configurations of individual orderings, see Hansson (1969).

3. Guha (1972) and Blau (1976) have shown that a binary SDR $f : T^l \mapsto C$ satisfying the Pareto-criterion yields quasi-transitive social weak preference relation for every $<R_i> \in T^l$ iff relative to every $(N - A)$, $A \subset N$, there is an oligarchy.

4. It can be shown that a neutral and monotonic binary social decision rule $f : T^l \mapsto C$ satisfying the Pareto-criterion yields acyclic social weak preference relation for every $<R_i> \in T^l$ iff for every $A \subset N$, every nonempty collection $\{V_1, V_2, ..., V_m\}$ of $(N - A)$-decisive sets has nonempty intersection, where $m \leq \#S$.

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