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Semi-Strict Majority Rules : Necessary and Sufficient Conditions for Quasi-Transitivity and Transitivity

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Abstract

It is shown that for every semi-strict majority rule a necessary and sufficient condition for quasi-transitivity is that the condition of absence of unique extremal value or value-restriction holds over every triple of alternatives and for transitivity is that the condition of strongly antagonistic preferences or partial agreement or strict placement restriction holds over every triple of alternatives.

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Under the majority rule an alternative is declared to be socially better than another if and only if the number of people who prefer the former to the latter is more than half of the number of people who are concerned with respect to the two alternatives in question, and under the strict majority rule if and only if the number of individuals who prefer the former to the latter is more than half of the total number of individuals. Semi-strict majority rules are more demanding than the majority rule but less stringent than the strict majority rule. The requirement in the case of a semi-strict majority rule for declaring an alternative to be socially better than another is that the number of people who prefer the former to the latter is more than half of some specified convex combination of the number of people who are nonindifferent over the two alternatives and the total number of individuals.

Inada [5] and Sen and Pattanaik [8] have obtained necessary and sufficient conditions for quasi-transitivity and transitivity of the social preference relation generated by the majority rule. They have shown that for quasi-transitivity a necessary and sufficient condition is that over every triple of alternatives value-restriction or limited agreement or extremal restriction holds. For transitivity, necessity and sufficiency of extremal restriction has been established. In [6,3] necessary and sufficient conditions for quasi-transitivity and transitivity of strict majority rule have been derived. A necessary and sufficient condition for quasi-transitivity of the social preference relation generated by the strict majority rule is that the Latin Square unique value restriction holds over every triple of alternatives. The satisfaction of the condition of conflictive preferences or extreme - value restriction over every triple of alternatives constitutes a necessary and sufficient condition for the transitivity of the social preference relation generated by the strict majority rule.

This paper is concerned with the derivation of necessary and sufficient conditions for quasi-transitivity and transitivity of semi-strict majority rules. We show that, for every semi-strict majority rule a necessary and sufficient condition for quasi-transitivity is that the condition of absence of unique extremal value or value-restriction holds over every triple of alternatives and for transitivity is that the condition of strongly antagonistic preferences or partial agreement or strict placement restriction holds over every triple of alternatives.

Two interesting features of these results are worth noting. The conditions for all semi-strict majority rules are identical for quasi-transitivity as well as transitivity and they are different from the corresponding conditions for the majority rule and the strict majority rule. How close a rule is to the majority rule or the strict majority rule does not make any difference. The only thing that seems to matter is that the rule lies in between the two rules.

If every individual has dichotomous preferences then both majority and strict majority rules yield quasi-transitive social preferences. This would lead one to expect that semi-strict majority rules, being the product of combining the two rules, would also yield quasi-transitive social preferences if every individual's preferences are dichotomous. However, one obtains the paradoxical result that for no semi-strict majority rule is the condition of dichotomous preferences sufficient for quasi-transitivity. The conditions for transitivity also display a similar discontinuity.

Restrictions on Preferences

The set of social alternatives would be denoted by S. The cardinality n of S would be assumed to be finite and greater than 2. The set of individuals and the number of individuals are designated by L and N respectively. Each individual $i \in L$ is assumed to have an ordering R_i defined over S. The symmetric and asymmetric parts of R_i are denoted by I_i and P_i respectively. The social preference relation is denoted by R and its symmetric and asymmetric components by I and P respectively. $N(\ )$ would stand for the number of individuals holding the preferences specified in the parentheses, and N_k for the number of individuals holding the k-th preference ordering.

Semi-strict majority rules : $\forall x, y \in S : x R y \longleftrightarrow$
$\sim [N (y P_i x) > \frac{1}{2} [p [N (x P_i y) + N (y P_i x)] + (1-p) N]]$, where p is a fraction such that $0 < p < 1$.

An individual is defined to be concerned with respect to a triple iff he is not indifferent over every pair of alternatives belonging to the triple; otherwise he is unconcerned. For individual i, in the triple $\{x, y, z\}$, x is best iff $(x R_i y \wedge x R_i z)$; medium iff $(y R_i x R_i z \vee z R_i x R_i y)$; worst iff $(y R_i x \wedge z R_i x)$; uniquely best iff $(x P_i y \wedge x P_i z)$; uniquely medium iff $(y P_i x P_i z \vee z P_i x P_i y)$; and uniquely worst iff $(y P_i x \wedge z P_i x)$.

Now we define several restrictions which specify the permissible sets of individual orderings. All the restrictions are defined over triples of alternatives.

Value - Restriction (VR) : It holds over a triple iff there is an alternative in the triple such that all concerned individuals agree that it is not best or agree that it is not medium or agree that it is not worst.

Absence of Unique Extremal Value (AUEV): It holds iff there does not exist an alternative such that it is uniquely best in some R_i or there does not exist an alternative such that it is uniquely worst in some R_i. Formally, AUEV holds over $\{x, y, z\}$ iff $\sim [\exists$ distinct $a, b, c \in \{x, y, z\}$: $\exists i : (a P_i b \wedge a P_i c)] \vee \sim [\exists$ distinct $a, b, c \in \{x, y, z\}$: $\exists i : (b P_i a \wedge c P_i a)]$.

Strict Placement Restriction (SPR) : It is satisfied over a triple iff there exists (i) an alternative such that it is uniquely best in every concerned R_i or (ii) an alternative such that it is uniquely worst in every concerned R_i or (iii) an alternative such that it is uniquely medium in every concerned R_i or (iv) a pair of distinct alternatives such that every individual is indifferent between the alternatives of the pair, i.e., SPR holds over $\{x, y, z\}$ iff $\exists$ distinct $a, b, c \in \{x, y, z\}$ such that [$\forall$ concerned i: $(a P_i b \wedge a P_i c) \vee \forall$ concerned $i : (b P_i a \wedge c P_i a) \vee \forall$ concerned $i : (b P_i a P_i c \vee c P_i a P_i b) \vee \forall i : a I_i b$].

Partial Agreement (PA) : It is satisfied over a triple iff every individual is concerned with respect to the triple and there exists an alternative belonging to the triple which is considered to be best by all or considered worst by all, i.e., PA holds over $\{x, y, z\}$ iff $[\sim(\exists i : x\, I_i\, y\, I_i\, z) \wedge \exists$ distinct $a, b, c \in \{x, y, z\} : \forall i : (a\, R_i\, b \wedge a\, R_i\, c)] \vee [\sim(\exists i : x\, I_i\, y\, I_i\, z) \wedge \exists$ distinct $a, b, c \in \{x, y, z\} : \forall i : (b\, R_i\, a \wedge c\, R_i\, a)]$.

Strongly Antagonistic Preferences (SAP) : It holds over $\{x, y, z\}$ iff $\exists$ distinct $a, b, c \in \{x, y, z\}$ such that

$$[\forall i : (a\, P_i\, b\, P_i\, c \;\vee\; c\, P_i\, b\, P_i\, a \;\vee\; b\, P_i\, a\, I_i\, c)] \;\vee$$
$$[\forall i : (a\, P_i\, b\, P_i\, c \;\vee\; c\, P_i\, b\, P_i\, a \;\vee\; a\, I_i\, c\, P_i\, b)] .$$

Conditions for Quasi - Transitivity

Theorem 1 : For every semi-strict majority rule, a necessary and sufficient condition for the quasi-transitivity of the social preference relation is that the condition of absence of unique extremal value or value-restriction holds over every triple of alternatives.

Proof : <u>Sufficiency</u>

Sen [9] has shown that for every monotonic and neutral binary social decision rule value-restriction is a sufficient condition for the quasi-transitivity of the social preference relation. As every semi-strict majority rule satisfies the condition of independence of irrelevant alternatives, monotonicity and neutrality, the sufficiency of value-restriction follows as a corollary of Sen's theorem.

Suppose AUEV is satisfied over the triple $\{x, y, z\}$. Then the set of R_i over $\{x, y, z\}$ must be a subset of either π_1 or π_2 which are as follows :

π_1		π_2	
1.	$x \; P_i \; y \; I_i \; z$	1.	$x \; I_i \; y \; P_i \; z$
2.	$y \; P_i \; z \; I_i \; x$	2.	$y \; I_i \; z \; P_i \; x$
3.	$z \; P_i \; x \; I_i \; y$	3.	$z \; I_i \; x \; P_i \; y$
4.	$x \; I_i \; y \; I_i \; z$	4.	$x \; I_i \; y \; I_i \; z$

Consider the set of orderings π_1. Because of symmetry it is sufficient to show that $x \; P \; y$ and $y \; P \; z$ imply $x \; P \; z$.

$$x\,P\,y \longrightarrow N\,(x\,P_i\,y) > \frac{1}{2}p\,[\,N\,(x\,P_i\,y) + N\,(y\,P_i\,x)] + \frac{1}{2}\,(1-p)\,N$$

$$\longrightarrow N_1 > \frac{1}{2}p\,(N_1 + N_2) + \frac{1}{2}\,(1-p)\,N \qquad (1)$$

$$y\,P\,z \longrightarrow N_2 > \frac{1}{2}p\,(N_2 + N_3) + \frac{1}{2}\,(1-p)\,N \qquad (2)$$

Combining (1) and (2) we obtain

$$N_1 + N_2 > \frac{1}{2}p\,(N_1 + 2\,N_2 + N_3) + \frac{1}{2}\,(1-p)\,N + \frac{1}{2}\,(1-p)N$$

$$\longrightarrow N_1 > \frac{1}{2}p\,(N_1 + N_3) + \frac{1}{2}\,(1-p)\,N + \frac{1}{2}\,(1-p)\,(N_1 + N_3 + N_4 - N_2) \qquad (3)$$

$$(1) \longrightarrow N_1 + N_3 + N_4 > \frac{1}{2}p\,(N_1 + N_2 + N_3 + N_4) + \frac{1}{2}\,(1-p)\,N + (1-\frac{1}{2}p)\,(N_3 + N_4)$$

$$\longrightarrow N_1 + N_3 + N_4 > \frac{N}{2} \qquad (4)$$

From (3) and (4) we conclude

$$N_1 > \frac{1}{2}p\,(N_1 + N_3) + \frac{1}{2}\,(1-p)\,N$$

which implies that $x\,P\,z$ holds. Proof of the other case is similar and therefore would be omitted here.

<u>Necessity</u>

It can be easily checked that a set of R_i violates both VR and AUEV over a triple $\{x, y, z\}$ iff the set of R_i includes one of the following 9 sets of orderings, except for a formal interchange of alternatives:

(A)	$x P_i y P_i z$	(B)	$x P_i y P_i z$	(C)	$x P_i y P_i z$
	$y P_i z P_i x$		$y P_i z P_i x$		$y P_i z P_i x$
	$z P_i x P_i y$		$z P_i x I_i y$		$z I_i x P_i y$
(D)	$x P_i y P_i z$	(E)	$x P_i y P_i z$	(F)	$x P_i y P_i z$
	$y P_i z I_i x$		$y I_i z P_i x$		$y I_i z P_i x$
	$z P_i x I_i y$		$z I_i x P_i y$		$z P_i x I_i y$
(G)	$x P_i y P_i z$	(H)	$x P_i y I_i z$	(I)	$x P_i y I_i z$
	$y P_i z I_i x$		$y P_i z I_i x$		$y I_i z P_i x$
	$z I_i x P_i y$		$z I_i x P_i y$		$z I_i x P_i y$

Therefore, for proving the necessity of (VR v AUEV) it suffices to show that for each of the above 9 sets there exists an assignment of individuals which results in non-quasitransitive social preferences.

Take for (A), (B) and (C) $N_1 = N_2 = N_3 = \frac{N}{3}$, for (D) $M > \frac{1}{p}$, $N = (4-p)\,M + 1$, $N_1 = N_2 = M$ and $N_3 = (2-p)\,M + 1$, for (E) $M > \frac{1}{p}$, $N = (4-p)\,M + 1$, $N_1 = N_3 = M$ and $N_2 = (2-p)\,M + 1$, for (F) $N > \max\left[\frac{4}{p^2}, \frac{2p}{(1-p)(2-p)}\right]$, $N_1 = \frac{N}{2} - 1$, $N_2 = \frac{1}{2}(1-p)\,N + 1$ and $N_3 = \frac{1}{2}\,p\,N$, for (G) $N_1 = \frac{1-p}{2-p}\,N$ and $N_2 = N_3 = \frac{1}{2\,(2-p)}\,N$, for (H) $N > \frac{2\,(2-p)}{p\,(1-p)}$, $N_1 = \frac{1-p}{2-p}\,N$, $N_2 = \frac{N}{2} - 1$ and $N_3 = \frac{p}{2(2-p)}\,N + 1$ and for (I) $N > \frac{2\,(2-p)}{p\,(1-p)}$, $N_1 = \frac{p}{2\,(2-p)}\,N + 1$, $N_2 = \frac{N}{2} - 1$ and $N_3 = \frac{1-p}{2-p}\,N$. This results for (A), (C), (G) and (H) in $x P y \wedge y P z \wedge \sim x P z$, for (B) and (D) in $y P z \wedge z P x \wedge \sim y P x$ and for (E), (F) and (I) in $z P x \wedge x P y \wedge \sim z P y$.

Conditions for transitivity

Lemma 1 : A set of orderings over a triple $\{x, y, z\}$ violates all three restrictions SPR, PA and SAP iff it contains one of the following 13 sets of orderings, except for a formal interchange of alternatives :

(A) $x\, P_i\, y\, P_i\, z$
$y\, P_i\, z\, P_i\, x$

(B) $x\, P_i\, y\, P_i\, z$
$y\, I_i\, z\, P_i\, x$

(C) $x\, P_i\, y\, P_i\, z$
$z\, P_i\, x\, I_i\, y$

(D) $x\, P_i\, y\, P_i\, z$
$y\, P_i\, z\, I_i\, x$
$x\, I_i\, y\, I_i\, z$

(E) $x\, P_i\, y\, P_i\, z$
$z\, I_i\, x\, P_i\, y$
$x\, I_i\, y\, I_i\, z$

(F) $x\, P_i\, y\, P_i\, z$
$y\, P_i\, z\, I_i\, x$
$z\, I_i\, x\, P_i\, y$

(G) $x\, P_i\, y\, I_i\, z$
$y\, P_i\, z\, I_i\, x$
$x\, I_i\, y\, I_i\, z$

(H) $x\, P_i\, y\, I_i\, z$
$y\, P_i\, z\, I_i\, x$
$z\, P_i\, x\, I_i\, y$

(I) $x\, I_i\, y\, P_i\, z$
$y\, I_i\, z\, P_i\, x$
$x\, I_i\, y\, I_i\, z$

(J) $x\, I_i\, y\, P_i\, z$
$y\, I_i\, z\, P_i\, x$
$z\, I_i\, x\, P_i\, y$

(K) $x\, P_i\, y\, I_i\, z$
$x\, I_i\, y\, P_i\, z$
$x\, I_i\, y\, I_i\, z$

(L) $x\, P_i\, y\, I_i\, z$
$y\, P_i\, z\, I_i\, x$
$z\, I_i\, x\, P_i\, y$

(M) $x\, P_i\, y\, I_i\, z$
$y\, I_i\, z\, P_i\, x$
$z\, I_i\, x\, P_i\, y$

Proof : It can be easily checked that SPR is violated iff the set of R_i contains one of the following 8 sets of orderings, except for a formal interchange of alternatives:

(i) $x \, P_i \, y \, P_i \, z$
$y \, P_i \, z \, P_i \, x$

(ii) $x \, P_i \, y \, P_i \, z$
$y \, I_i \, z \, P_i \, x$

(iii) $x \, P_i \, y \, P_i \, z$
$z \, P_i \, x \, I_i \, y$

(iv) $x \, P_i \, y \, P_i \, z$
$y \, P_i \, z \, I_i \, x$

(v) $x \, P_i \, y \, P_i \, z$
$z \, I_i \, x \, P_i \, y$

(vi) $x \, P_i \, y \, I_i \, z$
$y \, P_i \, z \, I_i \, x$

(vii) $x \, I_i \, y \, P_i \, z$
$y \, I_i \, z \, P_i \, x$

(viii) $x \, P_i \, y \, I_i \, z$
$x \, I_i \, y \, P_i \, z$

(i), (ii) and (iii) are the same as (A), (B) and (C) respectively, so it suffices to consider the remaining five sets. (iv) would violate PA iff either an ordering in which z is not worst or an unconcerned ordering is included. With the inclusion of required ordering, excepting the case when $z \, P_i \, y \, P_i \, x$ is included, the set of R_i contains one of the sets (A) - (M). If $z \, P_i \, y \, P_i \, x$ is included then SAP is violated iff an ordering not already contained in the set is included, and with the required inclusion one of (A) - (M) is contained in the set of R_i. Analysis for the case (v) is similar to that of (iv). (vi) would violate PA iff an ordering in which z is not worst or an unconcerned ordering is included. With the inclusion of required ordering the set of R_i contains one of the sets (A) - (M). The demonstration for (vii) is similar to that of (vi). Finally consider (viii). It would violate PA only if an ordering in which x is not best or an unconcerned ordering is included. Except the cases of inclusion of $y P_i x P_i z$ and $y P_i z I_i x$, in each case one of the sets (A) - (M) is contained in the set of R_i. If $y P_i x P_i z$ or $y P_i z I_i x$ is included, then PA is violated iff an ordering in which z is not worst or an unconcerned ordering is included. The set of R_i contains one of the sets (A) - (M) with the inclusion of the required ordering. The proof of the lemma is completed by noting that each of the sets (A) - (M) violates all three restrictions.

Theorem 2 : For every semi-strict majority rule, a necessary and sufficient condition for transitivity of the social preference relation is that the condition of strongly antagonistic preferences or partial agreement or strict placement restriction holds over every triple of alternatives.

Proof : <u>Sufficiency</u>

Suppose transitivity is violated. Then for some $x, y, z \in S$ we must have $x \, R \, y \wedge y \, R \, z \wedge z \, P \, x$.

$$x \, R \, y \longrightarrow N \, (y \, P_i \, x) \leqslant \frac{1}{2} p \, [N(x \, P_i \, y) + N \, (y \, P_i \, x)] + \frac{1}{2} (1-p) \, N$$

$$\longrightarrow N \, (y \, P_i \, x) \leqslant \frac{1}{2} p \, [N_c - N \, (\text{concerned } i: x \, I_i \, y)] + \frac{1}{2} (1-p) \, [N_c + N_u] \, ,$$

where N_c = number of individuals concerned with respect to $\{x, y, z\}$ and N_u = number of individuals unconcerned with respect to $\{x, y, z\}$

$$\longrightarrow N \, (\text{concerned } i : x \, R_i \, y) \geqslant \frac{1}{2} N_c + \frac{1}{2} p \, N \, (\text{concerned } i: x \, I_i \, y) - \frac{1}{2} (1-p) \, N_u \qquad (1)$$

$$\longrightarrow N\ (x\ R_i\ y) \geqslant \frac{1}{2} N + \frac{1}{2} p\ N\ (\text{concerned } i:\ x\ I_i\ y) + \frac{1}{2} p\ N_u \qquad (2)$$

Similarly,

$$y\ R\ z \longrightarrow N\ (\text{concerned } i:\ y\ R_i\ z) \geqslant \frac{1}{2} N_c + \frac{1}{2} p\ N\ (\text{concerned } i:\ y\ I_i\ z) - \frac{1}{2}(1-p)\ N_u \qquad (3)$$

$$\longrightarrow N\ (y\ R_i\ z) \geqslant \frac{1}{2} N + \frac{1}{2} p\ N\ (\text{concerned } i:\ y\ I_i\ z) + \frac{1}{2} p\ N_u \qquad (4)$$

$$z\ P\ x \longrightarrow N\ (z\ P_i\ x) > \frac{1}{2} N_c - \frac{1}{2} p\ N\ (\text{concerned } i:\ x\ I_i\ z) + \frac{1}{2}(1-p)\ N_u$$

$$\longrightarrow N\ (\text{concerned } i:\ z\ R_i\ x) > \frac{1}{2} N_c + (1 - \frac{1}{2} p)\ N\ (\text{concerned } i:\ x\ I_i\ z) + \frac{1}{2}(1-p)\ N_u \qquad (5)$$

$$\longrightarrow N\ (z\ R_i\ x) > \frac{1}{2} N + (1 - \frac{1}{2} p)\ N\ (\text{concerned } i:\ x\ I_i\ z) + \frac{1}{2}(2 - p)\ N_u \qquad (6)$$

$$(1) \longrightarrow N\ (\text{concerned } i:\ x\ R_i\ y) \geqslant \frac{1}{2} N_c - \frac{1}{2}(1-p)\ N_u \qquad (7)$$

$$(3) \longrightarrow N\ (\text{concerned } i:\ y\ R_i\ z) \geqslant \frac{1}{2} N_c - \frac{1}{2}(1-p)\ N_u \qquad (8)$$

$$(5) \longrightarrow N\ (\text{concerned } i:\ z\ R_i\ x) > \frac{1}{2} N_c + \frac{1}{2}(1-p)\ N_u \qquad (9)$$

$(8) \wedge (9) \longrightarrow \exists$ concerned $i : y R_i z R_i x$ (10)

$(7) \wedge (9) \longrightarrow \exists$ concerned $i : z R_i x R_i y$ (11)

$z P x \longrightarrow \exists i : z P_i x$ (12)

(10), (11) and (12) imply that SPR is violated.

Now,

$[(2) \wedge \exists$ concerned $i : x I_i y] \longrightarrow N (x R_i y) > \frac{N}{2}$ (13)

$[(4) \wedge \exists$ concerned $i : y I_i z] \longrightarrow N (y R_i z) > \frac{N}{2}$ (14)

From (13) we obtain

$(2) \wedge (4) \wedge \exists$ concerned $i : x I_i y \longrightarrow \exists i : x R_i y R_i z$ (15)

From (14) we obtain

$(2) \wedge (4) \wedge \exists$ concerned $i : y I_i z \longrightarrow \exists i : x R_i y R_i z$ (16)

$(15) \longrightarrow [\sim (\exists i : x P_i y) \longrightarrow (N_u \neq 0 \vee \exists i : x P_i z)]$ (17)

$(16) \longrightarrow [\sim (\exists i : y P_i z) \longrightarrow (N_u \neq 0 \vee \exists i : x P_i z)]$ (18)

From (17) and (18) we conclude

$$[N_u = 0 \wedge \sim (\exists i : x P_i z)] \longrightarrow [\exists i : x P_i y \wedge \exists i : y P_i z] \quad (19)$$

(19) is equivalent to

$$[N_u \neq 0 \vee \exists i : x P_i z \vee (\exists i : x P_i y \wedge \exists i : y P_i z)] \quad (20)$$

(10), (11) and (20) imply that PA is violated.

Suppose SAP holds. Then from (10) and (11) we conclude that ($\exists$ concerned i : $x I_i y$ v $\exists$ concerned i: $y I_i z$). By (15) and (16), this implies that $\exists i$: $x R_i y R_i z$. But this contradicts our supposition that SAP holds, in view of (10) and (11). Therefore SAP must be violated.

<u>Necessity</u>

By lemma 1, a set of R_i over a triple $\{x, y, z\}$ violates all three restrictions SPR, PA and SAP iff it includes one of the 13 sets (A) - (M) of lemma 1, except for a formal interchange of alternatives. Therefore, for proving the necessity of (SPR v PA v SAP) it suffices to show that for each of the 13 sets there exists an assignment of individuals which results in intransitive social preferences. Take for (A), (B) and (C) $N_1 = N_2 = \frac{N}{2}$, for (D) and (E) $N \geqslant \frac{(2-p)^2}{1-p}$, $N_1 = \frac{(1-p)N}{2(2-p)}$, N_2 =

$\frac{(1-p)N}{2(2-p)} + 1$ and $N_3 = \frac{N}{2-p} - 1$, for (F) $N_1 = \frac{(1-p)N}{2-p}$, $N_2 = \frac{p\ N}{2(2-p)}$ and $N_3 = \frac{N}{2}$, for (G) and (I) $N \geqslant \frac{2(2-p)}{p(1-p)}$, $N_1 = \frac{(1-p)\ N}{2-p} + 1$, $N_2 = \frac{(1-p)\ N}{2-p} - 1$ and $N_3 = \frac{p\ N}{2-p}$, for (H) and (J) $N \geqslant \frac{2\ (2+p)}{p\ (1-p)}$, $N_1 = \frac{N}{2+p} + 1$, $N_2 = \frac{p\ N}{2+p}$ and $N_3 = \frac{N}{2+p} - 1$, for (K) $N \geqslant \frac{2\ (2-p)}{1-p}$, $N_1 = \frac{(1-p)N}{2(2-p)} + 1$, $N_2 = \frac{(1-p)\ N}{2(2-p)}$ and $N_3 = \frac{N}{2-p} - 1$, for (L) $N_1 = \frac{(1-p)\ N}{2-p}$ and $N_2 = N_3 = \frac{N}{2(2-p)}$ and for (M) $N_1 = N_2 = \frac{N}{2(2-p)}$ and $N_3 = \frac{(1-p)\ N}{2-p}$. This results, for (C), (E), (F), (H), (L) and (M) in $x\ P\ y \wedge y\ I\ z \wedge x\ I\ z$, for (A), (B), (D) and (I) in $x\ I\ y \wedge y\ P\ z \wedge x\ I\ z$ and for (G), (J) and (K) in $x\ I\ y \wedge y\ I\ z \wedge x\ P\ z$.

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