Structure of Social Decision Rules Which are Simple Games

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I N D I A

ABSTRACT

The purpose of this paper is to study certain aspects of the structure of social decision rules which are simple games. The following characterization theorems have been proved in the paper : (1) A social decision rule (SDR) is a simple game iff it satisfies the conditions of (i) independence of irrelevant alternatives (ii) neutrality and (iii) monotonicity, and its structure is such that (iv) a coalition is blocking iff it is strictly blocking. (2) An SDR is a strong simple game iff it satisfies properties (i) - (iv) and its structure is such that (v) a coalition is blocking iff it is winning. (3) An SDR which is a simple game yields a social ordering for every profile of individual orderings iff it null or dictatorial. An Inada-type necessary and sufficient condition for transitivity under the class of SDRs which are non-dictatorial strong simple games is that the condition of weak Latin Square extremal value restriction holds for every triple of alternatives. An Inada-type necessary and sufficient condition for transitivity under the class of SDRs which are non-null non-strong simple games is that the condition of Latin Square extremal value restriction holds for every triple of alternatives. (4) An SDR which is a simple game yields a quasi-transitive social binary relation for every profile of individual orderings iff it null or oligarchic. An Inada-type necessary and sufficient condition for quasi-transitivity under the class of SDRs which are non-null non-oligarchic simple games is that the condition of Latin Square unique value restriction holds for every triple of alternatives.

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Structure of Social Decision Rules Which are Simple Games

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The purpose of this paper is to investigate certain aspects of the structure of social decision rules which are simple games. A simple game social decision rule (SDR) is defined by the condition that a social alternative x is preferred to another social alternative y iff all members of some winning coalition unanimously prefer x to y. A simple game is strong iff whenever a group of individuals is not a winning coalition, its complement constitutes a winning coalition. We show that a social decision rule is a simple game iff it satisfies the conditions of (i) independence of irrelevant alternatives (ii) neutrality and (iii) monotonicity, and its structure is such that (iv) a coalition is blocking iff it is strictly blocking. We also show that a social decision rule is a strong simple game iff it satisfies properties (i) - (iv) mentioned above and its structure is such that (v) a coalition is blocking iff it is winning.

Let S and N be the set of social alternatives and the set of individuals respectively, and l = #N. Let T and Q be the set of orderings of S and the set of reflexive, connected and quasi-transitive binary relations on S respectively. In the context of social decision rules which do not yield a rational (transitive, quasi-transitive or acyclic) social binary relation for every profile of individual orderings or for every profile of individual reflexive, connected and quasi-transitive binary relations, an important problem is that of characterizing subsets D of T or of Q such that for every profile of individual orderings belonging to D^l or for every profile of individual reflexive, connected and quasi-transitive binary relations belonging to D^l the social binary relation is rational. For several classes of social decision rules partial or complete characterization of subsets D of T or of Q have been obtained such that every profile of individual binary relations belonging to D^l necessarily gives rise to a rational social binary relation [see Inada (1969, 1970), Sen and Pattanaik (1969), Dummett and Farquharson (1961), Sen (1970), Pattanaik (1970, 1971), Fine (1973), and Jain (1984, 1986) among others]. In this paper we obtain, for the class of social decision rules which are simple games, complete characterization of (i) subsets D of the set of orderings T of the set of social alternatives S such that every profile of individual orderings belonging to D^l yields a social ordering and (ii) subsets D of the set of orderings T of the set of social alternatives S such that every profile of individual orderings belonging to D^l yields a reflexive, connected and quasi-transitive social binary relation.

For the purpose of obtaining complete characterization of subsets D of the set of orderings T of the set of social alternatives S such that every profile of individual orderings belonging to D^l yields a social ordering, we partition the class of social decision rules which are simple games into three sub-classes : (i) SDRs which are null or dictatorial simple games, (ii) SDRs which are non-dictatorial strong simple games and (iii) SDRs which are non-null non-strong simple games. We show that : (i) an SDR which is a simple game yields a social ordering for every profile of individual orderings iff it is null or dictatorial, and an SDR which is a strong simple game yields a social ordering for every profile of individual orderings iff it is dictatorial; (ii) an SDR which is a non-dictatorial strong simple game yields a social ordering for every profile of individual orderings belonging to D^l iff D satisfies the condition of weak Latin Square extremal value restriction (WLSEVR); and (iii) an SDR which is a non-null non-strong simple game yields a social ordering for every profile of individual orderings belonging to D^l iff D satisfies the condition of Latin Square extremal value restriction (LSEVR).

A social decision rule which is a simple game yields a reflexive, connected and quasi-transitive social binary relation for every profile of individual orderings iff it is null or oligarchic. We show that a non-null non-oligarchic simple game social decision rule yields a reflexive, connected and quasi-transitive social binary relation for every profile of individual orderings belonging to D^l iff D satisfies the condition of Latin Square unique value restriction (LSUVR). All three characterizing conditions, weak Latin Square extremal value restriction, Latin Square extremal value restriction and Latin Square unique value restriction are defined for triples of alternatives. A subset D of T is said to satisfy one of these conditions iff D satisfies the condition for every triple of alternatives.

1. **Notation and Definitions**

The set of social alternatives and the finite set of individuals constituting the society are denoted by S and N respectively. We assume the cardinality of S to be at least 3. We denote the cardinality of N by l and assume $l \geq 2$. Each individual $i \in N$ is assumed to have a binary weak preference relation R_i on S. We denote asymmetric parts of binary relations R_i, R'_i, R, R' etc., by P_i, P'_i, P, P' etc., respectively; and symmetric parts by I_i, I'_i, I, I' etc., respectively.

We define a binary relation R on a set S to be (i) reflexive iff $(\forall x \in S)$ (xRx), (ii) connected iff $(\forall x,y \in S)$ $(x \neq y \rightarrow xRy \vee yRx)$, (iii) quasi-transitive iff $(\forall x,y,z \in S)$ $(xPy \wedge yPz \rightarrow xPz)$, (iv) transitive iff $(\forall x,y,z \in S)$ $(xRy \wedge yRz \rightarrow xRz)$, and (v) an ordering iff it is reflexive, connected and transitive.

We denote by C the set of all reflexive and connected binary relations on S and by T the set of all reflexive, connected and transitive binary relations (orderings) on S. A social decision rule (SDR) f is a function from T^l to C; $f : T^l \mapsto C$. Thus, by definition, the domain of SDR will be taken to be the set of all logically possible l-tuples $(R_1,....,R_l)$ of individual orderings. Profiles $(R_1,....,R_l)$, $(R'_1,....,R'_l)$ etc., will be written as $<R_i>$, $<R'_i>$ etc., respectively in abbreviated form. The social weak preference relations corresponding to $<R_i>$, $<R'_i>$ etc., will be denoted by R, R' etc., respectively.

An SDR satisfies (i) weak Pareto-criterion (WP) iff $(\forall <R_i> \in T^l)$ $(\forall x,y \in S)$ $[(\forall i \in N)(xP_iy) \rightarrow xPy]$, (ii) independence of irrelevant alternatives (I) iff $(\forall <R_i>,<R'_i> \in T^l)$ $(\forall x,y \in S)$ $[(\forall i \in N)[(xR_iy \leftrightarrow xR'_iy) \wedge (yR_ix \leftrightarrow yR'_ix)] \rightarrow [(xRy \leftrightarrow xR'y) \wedge (yRx \leftrightarrow yR'x)]]$, and (iii) monotonicity (M) iff $(\forall <R_i>,<R'_i> \in T^l)$ $(\forall x \in S)$ $[(\forall i \in N)[(\forall a,b \in S - \{x\})(aR_ib \leftrightarrow aR'_ib) \wedge (\forall y \in S - \{x\})[(xP_iy \rightarrow xP'_iy) \wedge (xI_iy \rightarrow xR'_iy)]] \rightarrow (\forall y \in S - \{x\})[(xPy \rightarrow xP'y) \wedge (xIy \rightarrow xR'y)]]$.

Let Φ be the set of all permutations of the set of alternatives S. Let $\phi \in \Phi$. Corresponding to a binary relation R on a set S, we define the binary relation $\phi(R)$ on S by; $(\forall x,y \in S)$ $[\phi(x)\, \phi(R)\, \phi(y) \leftrightarrow xRy]$. An SDR satisfies neutrality (NT) iff $(\forall <R_i>,<R'_i> \in T^l)$ $[(\exists \phi \in \Phi)(\forall i \in N)[R'_i = \phi(R_i)] \rightarrow R' = \phi(R)]$.

It is clear from the definitions of conditions I, M and NT that an SDR $f : T^l \mapsto C$ satisfying condition I satisfies (i) neutrality iff $(\forall <R_i>,<R'_i> \in T^l)$ $(\forall x,y,z,w \in S)$ $[(\forall i \in N)[(xR_iy \leftrightarrow zR'_iw) \wedge (yR_ix \leftrightarrow wR'_iz)] \rightarrow [(xRy \leftrightarrow zR'w) \wedge (yRx \leftrightarrow wR'z)]]$, and (ii) monotonicity iff $(\forall <R_i>,<R'_i> \in T^l)$ $(\forall x,y \in S)$ $[(\forall i \in N)[(xP_iy \rightarrow xP'_iy) \wedge (xI_iy \rightarrow xR'_iy)] \rightarrow [(xPy \rightarrow xP'y) \wedge (xIy \rightarrow xR'y)]]$.

An SDR is called (i) null iff $(\forall <R_i> \in T^l)$ $(\forall x,y \in S)$ (xIy), (ii) dictatorial iff $(\exists j \in N)$ $(\forall <R_i> \in T^l)$ $(\forall x,y \in S)$ $(xP_jy \rightarrow xPy)$, and (iii) oligarchic iff $(\exists V \subset N)$ $(\forall <R_i> \in T^l)$ $(\forall x,y \in S)$ $[[(\forall i \in V)(xP_iy) \rightarrow xPy] \wedge [(\forall i \in V)(xP_iy \rightarrow xRy)]]$.

A coalition is a nonempty subset of N. A coalition V is defined to be winning iff $(\forall <R_i> \in T^l)$ $(\forall x,y \in S)$ $[(\forall i \in V)(xP_iy) \rightarrow xPy]$. We denote by W the set of all winning coalitions. $V \subset N$ is a minimal winning coalition iff V is a winning coalition and no proper subset of V is a winning coalition. The set of all minimal winning coalitions will be denoted by W_m. We define a coalition $V \subset N$ to be blocking iff $(\forall <R_i> \in T^l)$ $(\forall x,y \in S)$ $[(\forall i \in V)(xP_iy) \rightarrow xRy]$, and to be strictly blocking iff $(\forall <R_i> \in T^l)$ $(\forall x,y \in S)$ $[(\forall i \in V)(xR_iy) \rightarrow xRy]$. The set of all blocking coalitions will be denoted by B and the set of all strictly blocking coalitions by B_s.

Remark 1 : Consider an SDR f: $T^l \mapsto C$. If $V_1, V_2 \in W$ then $V_1 \cap V_2$ must be non-empty, because $V_1 \cap V_2 = \emptyset$ would lead to a contradiction if we have for $x,y \in S$, $[(\forall i \in V_1)(xP_iy) \wedge (\forall i \in V_2)(yP_ix)]$, giving rise to $(xPy \wedge yPx)$.
Remark 2 : Let $V \in W$. Then by the finiteness of V and the fact that the empty set can never be winning, it follows that there exists a nonempty $V' \subset V$ such that $V' \in W_m$.
Remark 3 : From the definitions of winning coalition, blocking coalition and strictly blocking coalition, it follows that if a coalition is winning or strictly blocking then it is blocking.

A social decision rule is defined to be (i) a simple game iff $(\forall <R_i> \in T^l)$ $(\forall x,y \in S)$ $[xPy \leftrightarrow (\exists V \in W)(\forall i \in V)(xP_iy)]$, and (ii) a strong simple game iff it is a simple game and $(\forall V \subset N)$ $[V \notin W \rightarrow (N - V) \in W]$.

2. Characterization of Social Decision Rules which are Simple Games

Theorem 1 : A social decision rule f: $T^l \mapsto C$ is a simple game iff it satisfies the conditions of (i) independence of irrelevant alternatives (ii) neutrality and (iii) monotonicity, and (iv) its structure is such that a coalition is blocking iff it is strictly blocking[1].

Proof : Let SDR f be a simple game.

Consider any $<R_i>$, $<R'_i> \in T^l$ and any $x, y \in S$ such that $(\forall i \in N)\ [(xR_iy \leftrightarrow xR'_iy) \wedge (yR_ix \leftrightarrow yR'_i\,x)]$.

$$
\begin{array}{lll}
xPy & \rightarrow & (\exists V \in W)\ (\forall i \in V)\ (xP_iy) \\
 & \rightarrow & (\forall i \in V)\ (xP'_iy) \\
 & \rightarrow & xP'y, \\
xP'y & \rightarrow & (\exists V' \in W)\ (\forall i \in V')\ (xP''_iy) \\
 & \rightarrow & (\forall i \in V')\ (xP_iy) \\
 & \rightarrow & xPy.
\end{array}
$$

Thus $(xPy \leftrightarrow xP'y)$. By an analogous argument we obtain $(yPx \leftrightarrow yP'x)$. $[(xPy \leftrightarrow xP'y) \wedge (yPx \leftrightarrow yP'x)]$ implies $(xIy \leftrightarrow xI'y)$, in view of the fact that R and R′ are connected. This establishes that f satisfies condition I.

Now consider any $<R_i>$, $<R'_i> \in T^l$ and any $x,y,z,w \in S$ such that $(\forall i \in N)\ [(xR_iy \leftrightarrow zR'_iw) \wedge (yR_ix \leftrightarrow wR'_iz)]$. Designate by N_1, N_2 and N_3 the sets $\{i \in N \mid xP_iy \wedge zP'_iw\}$, $\{i \in N \mid xI_iy \wedge zI'_iw\}$ and $\{i \in N \mid yP_ix \wedge wP'_iz\}$ respectively. $xPy \rightarrow (\exists V \subset N_1)\ (V \in W)$, which in turn implies $zP'w$. Similarly $zP'w \rightarrow xPy$. Thus $(xPy \leftrightarrow zP'w)$. By an analogous argument we obtain $(yPx \leftrightarrow wP'z)$. As R and R′ are connected it follows that $(xIy \leftrightarrow zI'w)$, in view of the fact that $[(xPy \leftrightarrow zP'w) \wedge (yPx \leftrightarrow wP'z)]$. This establishes that f is neutral.

Next consider any $<R_i>$,$<R'_i> \in T^l$ and any $x,y \in S$ such that $(\forall i \in N)\ [(xP_iy \rightarrow xP'_iy) \wedge (xI_iy \rightarrow xR'_iy)]$. Designate by N_1, N_2 and N_3 the sets $\{i \in N \mid xP_iy\}$, $\{i \in N \mid xI_iy\}$ and $\{i \in N \mid yP_ix\}$ respectively; and by N'_1, N'_2 and N'_3 the sets $\{i \in N \mid xP'_iy\}$, $\{i \in N \mid xI'_iy\}$ and $\{i \in N \mid yP'_ix\}$ respectively.

$$
\begin{array}{lll}
xPy & \rightarrow & (\exists V \subset N_1)\ (V \in W) \\
 & \rightarrow & (\forall i \in V)\ (xP'_iy), \text{ as } N_1 \subset N'_1 \\
 & \rightarrow & xP'y, \\
yP'x & \rightarrow & (\exists V' \subset N'_3)\ (V' \in W) \\
 & \rightarrow & (\forall i \in V')\ (yP_ix), \text{ as } N''_3 \subset N_3 \\
 & \rightarrow & yPx.
\end{array}
$$

$(yP'x \rightarrow yPx)$ is equivalent to $(xRy \rightarrow xR'y)$, as R and R′ are connected. $(xPy \rightarrow xP'y)$ and $(xRy \rightarrow xR'y)$ establish that f is monotonic.

If V is a strictly blocking coalition then V is a blocking coalition. Suppose V is not a strictly blocking coalition. Then $(\exists <R'_i> \in T^l)\ (\exists x,y \in S)\ [(\forall i \in V)\ (xR'_iy) \wedge yP'x]$.

$[(\forall i \in V)\ (xR'_iy) \wedge yP'x] \rightarrow [(\exists V' \subset N - V)\ (V' \in W)]$

$\rightarrow (\forall <R_i> \in T^l)\ [(\forall i \in V)\ (xP_iy) \wedge (\forall i \in N - V)\ (yP_ix) \rightarrow yPx]$

$\rightarrow$ V is not a blocking coalition.

Thus a coalition is blocking iff it is strictly blocking.

We have shown that if SDR f is a simple game then it satisfies the conditions of (i) independence of irrelevant alternatives (ii) neutrality and (iii) monotonicity, and (iv) its structure is such that a coalition is blocking iff it is strictly blocking.

Now let f be a social decision rule such that it satisfies the conditions of (i) independence of irrelevant alternatives (ii) neutrality and (iii) monotonicity, and (iv) its structure is such that a coalition is blocking iff it is strictly blocking. Consider any situation $<R'_i> \in T^l$ and any $x,y \in S$ such that $xP'y$. By condition I, $xP'y$ is a consequence solely of individual preferences over $\{x,y\}$ in $<R'_i>$ situation. Designate by N_1, N_2 and N_3 the sets $\{i \in N \mid xP'_iy\}$, $\{i \in N \mid xI'_iy\}$ and $\{i \in N \mid yP'_ix\}$ respectively. Now consider any $<R''_i> \in T^l$ such that $[(\forall i \in N_1)\ (xP''_iy) \wedge (\forall i \in N_2 \cup N_3)\ (yP''_ix)]$. Suppose $yR''x$. Then by conditions I, NT and M we conclude $(\forall <R_i> \in T^l)\ (\forall a,b \in S)\ [(\forall i \in N_2 \cup N_3)\ (aP_ib) \rightarrow aRb]$. In other words, $N_2 \cup N_3$ is a blocking coalition. As every blocking coalition is strictly blocking, it follows that $N_2 \cup N_3$ is strictly blocking. Consequently we must have $yR'x$ as $(\forall i \in N_2 \cup N_3)\ (yR'_ix)$. This, however, contradicts the hypothesis $xP'y$. Therefore, we conclude that $[(\forall i \in N_1)\ (xP''_iy) \wedge (\forall i \in N_2 \cup N_3)\ (yP''_ix)]$ must result in $xP''y$. Then it follows by conditions I, NT and M, that $(\forall <R_i> \in T^l)\ (\forall a,b \in S)\ [(\forall i \in N_1)\ (aP_ib) \rightarrow aPb]$. That is to say, N_1 is a winning coalition. We have shown that $(\forall <R_i> \in T^l)\ (\forall a,b \in S)\ [aPb \rightarrow (\exists V \in W)\ (\forall i \in V)\ (aP_ib)]$. This coupled with the fact that if $V \in W$, then $(\forall <R_i> \in T^l)\ (\forall a,b \in S)\ [(\forall i \in V)\ (aP_ib) \rightarrow aPb]$, establishes the fact that f is a simple game.

3. Characterization of Social Decision Rules which are Strong Simple Games

Theorem 2 : A social decision rule f: $T^l \mapsto C$ is a strong simple game iff it satisfies the conditions of (i) independence of irrelevant alternatives, (ii) neutrality and (iii) monotonicity, and its structure is such that (iv) a coalition is blocking iff it is strictly blocking and (v) a coalition is blocking iff it is winning.

Proof : Let f be a strong simple game. Then by Theorem 1 it follows that f satisfies conditions (i) - (iv). Suppose V is a blocking coalition. Then N − V cannot be a winning coalition. By the definition of a strong simple game then it follows that the complement of N − V, i.e., V must be winning. This coupled with the fact that every winning coalition is blocking establishes that (v) holds.

Next let SDR f satisfy (i) - (v). From Theorem 1 we know that (i) - (iv) imply that f is a simple game. Suppose $V \subset N$ is not winning. Then $(\exists <R'_i> \in T^l)$ $(\exists x,y \in S)$ $[(\forall i \in V)\ (xP'_i y) \wedge yR'x]$. By conditions I, NT and M then it follows that $(\forall <R_i> \in T^l)$ $(\forall a,b \in S)$ $[(\forall i \in N - V)\ (aP_i b) \rightarrow aRb]$. In other words, N − V is a blocking coalition. As every blocking coalition is winning, it follows that N − V is winning. Thus we have shown that $(\forall V \subset N)\ [V \notin W \rightarrow (N - V) \in W]$, which establishes that f is a strong simple game.

4. Restrictions on Preferences

Let $A \subset S$ and let R be a binary relation on S. We define the restriction of R to A, denoted by R|A, by $R|A = R \cap (A \times A)$. Let $D \subset T$. We define the restriction of D to A, denoted by D|A, by $D|A = \{R|A \mid R \in D\}$.

A set of three distinct alternatives will be called a triple of alternatives. Let R be an ordering of S and let A be a triple of alternatives such that $A \subset S$. We define R to be unconcerned over A iff $(\forall x,y \in A)$ (xIy). R is said to be concerned over A iff it is not unconcerned over A.

Let $A = \{x,y,z\} \subset S$ be a triple of alternatives and let R be an ordering over S. We define in A, according to R, x to be best iff $(xRy \wedge xRz)$, to be medium iff $(yRxRz \vee zRxRy)$, and to be worst iff $(yRx \wedge zRx)$.

Weak Latin Square (WLS) : Let $A = \{x,y,z\} \subset S$ be a triple of alternatives and let R^s, R^t, R^u be orderings over S. The set $\{R^s|A, R^t|A, R^u|A\}$ forms a weak Latin Square over A iff ($\exists$ distinct $a,b,c \in A$) $[aR^s bR^s c \wedge bR^t cR^t a \wedge cR^u aR^u b]$.

The above weak Latin Square will be denoted by WLS(abca).

Remark 4 : $R^s|A$, $R^t|A$, $R^u|A$ in the definition of weak Latin Square need not be distinct. {xIyIz} forms a weak Latin Square over the triple {x,y,z}.

Latin Square (LS) : Let $A = \{x,y,z\} \subset S$ be a triple of alternatives and let R^s, R^t, R^u be orderings of S. The set $\{R^s|A, R^t|A, R^u|A\}$ of orderings forms a Latin Square over A iff R^s, R^t, R^u are concerned over A and ($\exists$ distinct $a,b,c \in A$) $[aR^s bR^s c \wedge bR^t cR^t a \wedge cR^u aR^u b]$.

The above Latin Square will be denoted by LS(abca).

Remark 5 : From the definitions of weak Latin Square and Latin Square it is clear that if orderings $R^s|A$, $R^t|A$, $R^u|A$ are concerned over A then they form a Latin Square iff they form a weak Latin Square.

Let $A = \{x,y,z\} \subset S$ be a triple of alternatives. For any distinct $a,b,c \in A$ we define :

$T[WLS(abca)] = \{R \in T|A \mid (aRbRc \vee bRcRa \vee cRaRb)\}$

$T[LS(abca)] = \{R \in T|A \mid R \text{ is concerned over } A \wedge (aRbRc \vee bRcRa \vee cRaRb)\}$

Thus we have :

T[WLS(xyzx)] = T[WLS(yzxy)] = T[WLS(zxyz)]

= {xPyPz, xPyIz, xIyPz, yPzPx, yPzIx, yIzPx, zPxPy, zPxIy, zIxPy, xIyIz},

T[WLS(xzyx)] = T[WLS(zyxz)] = T[WLS(yxzy)]

= {xPzPy, xPzIy, xIzPy, zPyPx, zPyIx, zIyPx, yPxPz, yPxIz, yIxPz, xIyIz},

T[LS(xyzx)] = T[LS(yzxy)] = T[LS(zxyz)] = T[WLS(xyzx)] − {xIyIz},

T[LS(xzyx)] = T[LS(zyxz)] = T[LS(yxzy)] = T[WLS(xzyx)] − {xIyIz}.

Now we define three restrictions on sets of orderings.

Latin Square Extremal Value Restriction (LSEVR) : Let $D \subset T$ be a set of orderings of S. Let $A = \{x,y,z\} \subset S$ be a triple of alternatives. D satisfies LSEVR over the triple A iff there do not exist distinct $a,b,c \in A$ and R^s, $R^t \in D|A \cap T[LS(abca)]$ such that (i) alternative a is uniquely best in R^s, and medium in R^t without being worst; and (ii) alternative b is uniquely worst in R^t, and medium in R^s without being best. More formally, $D \subset T$ satisfies LSEVR over the triple A iff $\sim [(\exists \text{ distinct } a,b,c \in A)\ (\exists R^s, R^t \in D|A \cap T[LS(abca)])\ (aP^s bR^s c \wedge cR^t aP^t b)]$. D satisfies LSEVR iff it satisfies LSEVR over every triple of alternatives contained in S.

Weak Latin Square Extremal Value Restriction[2] (WLSEVR) : Let $D \subset T$ be a set of orderings of S. Let $A = \{x,y,z\} \subset S$ be a triple of alternatives. D satisfies WLSEVR over the triple A iff there do not exist distinct a,b,c

$\in$ A and $R^s, R^t, R^u \in D|A \cap T[WLS(abca)]$ such that (i) R^s, R^t, R^u form WLS(abca), (ii) alternative a is uniquely best in R^s, and medium in R^t without being worst, and (iii) alternative b is uniquely worst in R^t, and medium in R^s without being best. More formally, $D \subset T$ satisfies WLSEVR over the triple A iff $\sim [(\exists$ distinct $a,b,c \in A) (\exists R^s, R^t, R^u \in D|A \cap T[WLS(abca)]) (aP^s bR^s c \wedge bR^u cR^u a \wedge cR^t aP^t b)]$. D satisfies WLSEVR iff it satisfies WLSEVR over every triple of alternatives contained in S.

Latin Square Unique Value Restriction (LSUVR) : Let $D \subset T$ be a set of orderings of S. Let $A = \{x,y,z\} \subset S$ be a triple of alternatives. D satisfies LSUVR over the triple A iff there do not exist distinct $a,b,c \in A$ and R^s, R^t, $R^u \in D|A \cap T[LS(abca)]$ such that (i) alternative b is uniquely medium in R^s, uniquely best in R^t, and uniquely worst in R^u; and (ii) R^s, R^t, R^u form LS(abca). More formally, $D \subset T$ satisfies LSUVR over the triple A iff $\sim [(\exists$ distinct $a,b,c \in A) (\exists R^s, R^t, R^u \in D|A \cap T[LS(abca)]) (aP^s bP^s c \wedge bP^t cR^t a \wedge cR^u aP^u b)]$. D satisfies LSUVR iff it satisfies LSUVR over every triple of alternatives contained in S.

5. Transitivity under Simple Games

Lemma 1 : Let social decision rule $f : T^l \mapsto C$ be a simple game. Then, f yields transitive social weak preference relation for every $\langle R_i \rangle \in T^l$ iff it is null or dictatorial.

Proof : If f is null then obviously $R = f\langle R_i \rangle$ is transitive for every $\langle R_i \rangle \in T^l$. If f is dictatorial then there is a minimal winning coalition consisting of a single individual, say individual j. As by definition, every winning coalition is blocking , it follows that {j} is a blocking coalition. By Theorem 1, if f is a simple game then every blocking coalition is strictly blocking. From the fact that {j} is both winning and strictly blocking we conclude that for every $\langle R_i \rangle \in T^l$, $R = f\langle R_i \rangle$ coincides with R_j. Transitivity of R follows from the fact that R_j is an ordering.

Now suppose f yields transitive R for every $\langle R_i \rangle \in T^l$. As f is a simple game, it satisfies conditions I, M and NT, by Theorem 1. If f satisfies the weak Pareto criterion then from Arrow's Impossibility Theorem it follows that f must be dictatorial. On the other hand, if the weak Pareto criterion is violated then f must be null as a consequence of conditions I, M and NT. This establishes the lemma.

Lemma 2 : Let social decision rule $f : T^l \mapsto C$ be a strong simple game. Then f yields transitive social weak preference relation for every $\langle R_i \rangle \in T^l$ iff f is dictatorial.

Proof : As f is a strong simple game, the set of all individuals N is winning. Therefore f cannot be null. The lemma now follows directly from Lemma 1.

Theorem 3 : Let social decision rule $f : T^l \mapsto C$ be a non-null non-strong simple game. Let $D \subset T$. Then f yields transitive social weak preference relation for every $\langle R_i \rangle \in D^l$ iff D satisfies the condition of Latin Square extremal value restriction.

Proof : Suppose $R = f\langle R_i \rangle$, $\langle R_i \rangle \in D^l$, violates transitivity. Then,

$(\exists x,y,z \in S) [xRy \wedge yRz \wedge zPx]$. (1)

$zPx \rightarrow (\exists V \in W) (\forall i \in V) (zP_i x)$, (2)

by the definition of a simple game;

$xRy \rightarrow (\exists j \in V) (xR_j y)$, (3)

as $(\forall i \in V) (yP_i x)$ would imply yPx, by the definition of a winning coalition;

$yRz \rightarrow (\exists k \in V) (yR_k z)$, (4)

as $(\forall i \in V) (zP_i y)$ would imply zPy, by the definition of a winning coalition;

$(2) \wedge (3) \rightarrow (\exists j \in V) (zP_j xR_j y)$ (5)

$(2) \wedge (4) \rightarrow (\exists k \in V) (yR_k zP_k x)$ (6)

(1) implies that x,y,z are distinct alternatives. $zP_j xR_j y$ and $yR_k zP_k x$ belong to T[LS(xyzx)]. In the triple {x,y,z}, z is uniquely best according to $zP_j xR_j y$, and medium according to $yR_k zP_k x$ without being worst; furthermore x is uniquely worst according to $yR_k zP_k x$ and medium according to $zP_j xR_j y$ without being best. Therefore LSEVR is violated over the triple {x,y,z}. Thus D violates LSEVR. We have shown that violation of transitivity by $R = f\langle R_i \rangle$, $\langle R_i \rangle \in D^l$, implies violation of LSEVR by D, which establishes the sufficiency of LSEVR for transitivity.

Suppose $D \subset T$ violates LSEVR. Then there exist distinct $x,y,z \in S$ such that D violates LSEVR over {x,y,z}. Violation of LSEVR by D over {x,y,z} implies $(\exists$ distinct $a,b,c \in \{x,y,z\}) (\exists R^s, R^t \in D) [aP^s bR^s c \wedge cR^t aP^t b]$. As f is non-null we conclude that N is a winning coalition. Because f is not a strong simple game, there exists a partition of N, (V, N − V), such that neither V nor N − V is a winning coalition. Now consider any $<R_i> \in D^l$ such that the restriction of $<R_i>$ to {x,y,z}, $<R_i|\{x,y,z\}>$, is given by : $[(\forall i \in V)\ (aP_i bR_i c) \wedge (\forall i \in N - V)\ (cR_i aP_i b)]$. In view of the fact that N is winning but neither V nor N − V is winning we conclude that $(aPb \wedge bIc \wedge aIc)$ holds, which violates transitivity. We have shown that if $D \subset T$ violates LSEVR then there exists $<R_i> \in D^l$ such that $R = f<R_i>$ is intransitive , i.e., if f yields transitive R for every $<R_i> \in D^l$ then D must satisfy LSEVR. This establishes the theorem.

Theorem 4 : Let social decision rule $f : T^l \mapsto C$ be a non-dictatorial strong simple game. Let $D \subset T$. Then f yields transitive social weak preference relation for every $<R_i> \in D^l$ iff D satisfies the condition of weak Latin Square extremal value restriction.

Proof : Suppose $R = f<R_i>$, $<R_i> \in D^l$, violates transitivity. Then,

$(\exists x,y,z \in S)\ [xRy \wedge yRz \wedge zPx]$. (1)

Designate by V_1, V_2 and V_3 the sets $\{i \in N \mid xR_i y\}$, $\{i \in N \mid yR_i z\}$, and $\{i \in N \mid zP_i x\}$ respectively.

$xRy \rightarrow N - V_1$ is not a winning coalition

$\rightarrow V_1$ is a winning coalition, by the definition of a strong simple game (2)

$yRz \rightarrow N - V_2$ is not a winning coalition

$\rightarrow V_2$ is a winning coalition, by the definition of a strong simple game (3)

$zPx \rightarrow V_3$ is a winning coalition, by the definition of a simple game (4)

As intersection of any two winning coalitions is non-empty (Remark 1), we conclude that :

$(\exists i \in N)\ (xR_i yR_i z)$, as $V_1 \cap V_2 \neq \emptyset$

$(\exists j \in N)\ (yR_j zP_j x)$, as $V_2 \cap V_3 \neq \emptyset$

$(\exists k \in N)\ (zP_k xR_k y)$, as $V_3 \cap V_1 \neq \emptyset$.

(1) implies that x,y,z are distinct alternatives. $xR_i yR_i z$, $yR_j zP_j x$ and $zP_k xR_k y$ form WLS(xyzx), and belong to T[WLS(xyzx)]. In the triple {x,y,z}, z is uniquely best according to $zP_k xR_k y$, and medium according to $yR_j zP_j x$ without being worst; furthermore x is uniquely worst according to $yR_j zP_j x$, and medium according to $zP_k xR_k y$ without being best. Therefore WLSEVR is violated over the triple {x,y,z}. Thus D violates WLSEVR. We have shown that violation of transitivity by $R = f<R_i>$, $<R_i> \in D^l$, implies violation of WLSEVR by D, which establishes the sufficiency of WLSEVR for transitivity.

Suppose $D \subset T$ violates WLSEVR. Then there exist distinct $x,y,z \in S$ such that D violates WLSEVR over {x,y,z}. Violation of WLSEVR by D over {x,y,z} implies $(\exists$ distinct $a,b,c \in \{x,y,z\})$ $(\exists R^s, R^t, R^u \in D)$ $[bR^u cR^u a \wedge cR^t aP^t b \wedge aP^s bR^s c]$. As f is a strong simple game, it follows that $N \in W$. Consequently the set of minimal winning coalitions W_m is nonempty. Let $V \in W_m$. As f is non-dictatorial, V must contain at least two individuals. Because f is a strong simple game it follows that $V \neq N$. Let (V_1, V_2) be a partition of V such that both V_1 and V_2 are non-empty. Consider any $<R_i> \in D^l$ such that the restriction of $<R_i>$ to {x,y,z}, $<R_i|\{x,y,z\}>$, is given by $[(\forall i \in V_1)\ (bR_i cR_i a) \wedge (\forall i \in V_2)\ (cR_i aP_i b) \wedge (\forall i \in N - V)\ (aP_i bR_i c)]$. By construction none of the sets, V_1, V_2, $N - V$, is a winning coalition. As f is a strong simple game, union of any two of the sets, V_1,V_2, $N - V$, is a winning coalition. By Theorem 2, if f is a strong simple game then a coalition is winning iff it is strictly blocking. So, $V_1 \cup V_2$, $V_1 \cup (N - V)$ and $V_2 \cup (N - V)$ are strictly blocking. In view of the fact that $V_1 \cup V_2$, $V_1 \cup (N - V)$, $V_2 \cup (N - V)$ are winning as well as strictly blocking and that none of the sets, V_1, V_2, $N - V$, is winning or strictly blocking, we conclude that $[aPb \wedge bRc \wedge cRa]$ holds, which violates transitivity. We have shown that if $D \subset T$ violates WLSEVR then there exists $<R_i> \in D^l$ such that $R = f<R_i>$ is intransitive, i.e., if f yields transitive R for every $<R_i> \in D^l$ then D must satisfy WLSEVR. This establishes the theorem.

6. Quasi-Transitivity under Simple Games

Lemma 3 : Let social decision rule $f : T^l \mapsto C$ be a simple game. Then, f yields quasi-transitive social weak preference relation for every $<R_i> \in T^l$ iff it is null or there is a unique minimal winning coalition.

Proof : If f is null then $R = f<R_i>$ is transitive for every $<R_i> \in T^l$. Suppose there is a unique minimal winning coalition V. Consider any $<R_i> \in T^l$ and any $x,y,z \in S$ such that $(xPy \wedge yPz)$ obtains. $xPy \rightarrow (\exists V' \in W)$ $(\forall i \in V')\ (xP_i y)$, by the definition of a simple game. Now it must be the case that $V \subset V'$, otherwise the fact that V is the unique minimal winning coalition will be contradicted. Consequently, $xPy \rightarrow (\forall i \in V)\ (xP_i y)$. By an analogous argument we obtain $[yPz \rightarrow (\forall i \in V)\ (yP_i z)]$. From $(\forall i \in V)\ (xP_i y \wedge yP_i z)$ we obtain $(\forall i \in V)$ $(xP_i z)$, which implies xPz. This proves that social weak preference relation is quasi-transitive for every $<R_i> \in T^l$.

Now suppose f yields quasi-transitive social weak preference relation for every $<R_i> \in T^l$. As f is a simple game it satisfies conditions I, NT and M, by Theorem 1. If the weak Pareto-criterion is satisfied then by Gibbard's Theorem [Gibbard (1969)] it follows that there must be a unique minimal winning coalition. On the other hand, if the weak Pareto-criterion is violated then f must be null as a consequence of conditions I, NT and M. This establishes the lemma.

Remark 6 : If SDR $f : T^l \mapsto C$ is a simple game then it satisfies conditions I, NT and M. Consequently there is a unique minimal winning coalition iff f is oligarchic. Therefore, Lemma 3 could be restated as follows :

Let SDR $f: T^l \mapsto C$ be a simple game. Then f yields quasi-transitive social weak preference relation for every $<R_i> \in T^l$ iff it is null or oligarchic.

Theorem 5 : Let social decision rule $f : T^l \mapsto C$ be a non-null non-oligarchic simple game. Let $D \subset T$. Then f yields quasi-transitive social weak preference relation for every $<R_i> \in D^l$ iff D satisfies the condition of Latin Square unique value restriction[3].

Proof : Suppose $R = f<R_i>$, $<R_i> \in D^l$, violates quasi-transitivity. Then,

$(\exists x,y,z \in S)\ [xPy \wedge yPz \wedge zRx]$ (1)

$xPy \rightarrow (\exists V_1 \in W)\ (\forall i \in V_1)\ (xP_iy),$ (2)

by the definition of a simple game

$yPz \rightarrow (\exists V_2 \in W)\ (\forall i \in V_2)\ (yP_iz),$ (3)

by the definition of a simple game

$(2) \wedge (3) \rightarrow (\exists i \in V_1 \cap V_2)\ (xP_iyP_iz)$, as $V_1 \cap V_2 \neq \emptyset$ by Remark 1

$zRx \rightarrow (\exists j \in V_2)\ (yP_jzR_jx)$, as $(\forall i \in V_2)\ (xP_iz)$ would imply xPz

$zRx \rightarrow (\exists k \in V_1)\ (zR_kxP_ky)$, as $(\forall i \in V_1)\ (xP_iz)$ would imply xPz.

(1) implies that x,y,z are distinct alternatives. xP_iyP_iz, yP_jzR_jx and zR_kxP_ky belong to T[LS(xyzx)], and form LS(xyzx). In the triple {x,y,z}, y is uniquely medium according to xP_iyP_iz; is uniquely best according to yP_jzR_jx; and is uniquely worst according to zR_kxP_ky. Therefore LSUVR is violated over the triple {x,y,z}. Thus D violates LSUVR. We have shown that violation of quasi-transitivity by $R = f<R_i>$, $<R_i> \in D^l$, implies violation of LSUVR by D, which establishes the sufficiency of LSUVR for quasi-transitivity.

Suppose $D \subset T$ violates LSUVR. Then there exist distinct $x,y,z \in S$ such that D violates LSUVR over the triple {x,y,z}. Violation of LSUVR by D over {x,y,z} implies $(\exists \text{ distinct } a,b,c \in \{x,y,z\})\ (\exists R^s, R^t, R^u \in D)$ $[aP^sbP^sc \wedge bP^tcR^ta \wedge cR^uaP^ub]$. As f is a non-null non-oligarchic simple game, it follows that there exist distinct $V_1, V_2 \in W$ such that V_1 and V_2 are minimal winning coalitions. $V_1 \cap V_2 \neq \emptyset$ follows from Remark 1, and $V_1 \cap V_2 \notin W$ from the fact that V_1 and V_2 are distinct minimal winning coalitions. Now consider any $<R_i> \in D^l$ such that the restriction of $<R_i>$ to {x,y,z}, $<R_i|\{x,y,z\}>$, is given by : $[(\forall i \in V_1 \cap V_2)\ (aP_ibP_ic) \wedge (\forall i \in V_1 - V_2)\ (bP_icR_ia) \wedge (\forall i \in N - V_1)\ (cR_iaP_ib)]$. $[V_2 \in W \wedge (\forall i \in V_2)\ (aP_ib) \rightarrow aPb]$ and $[V_1 \in W \wedge (\forall i \in V_1)\ (bP_ic) \rightarrow bPc]$. $\{i \in N \mid aP_ic\} = V_1 \cap V_2$ and $V_1 \cap V_2 \notin W$ imply cRa, as f is a simple game. $(aPb \wedge bPc \wedge cRa)$ implies that R violates quasi-transitivity. We have shown that if $D \subset T$ violates LSUVR then there exists $<R_i> \in D^l$ such that $R = f<R_i>$ violates quasi-transitivity, i.e., if f yields quasi-transitive R for every $<R_i> \in D^l$ then D must satisfy LSUVR. This establishes the theorem.

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Notes

1. For an alternative characterization of simple games see Bloomfield (1976).
2. Both Latin Square extremal value restriction and weak Latin Square extremal value restriction are weakened versions of extremal value restriction [see Jain (1984)]. A set of orderings D satisfies extremal value restriction over a triple of alternatives A iff (i) whenever an alternative is uniquely best in some ordering belonging to D|A, it is not medium in any ordering belonging to D|A unless it is worst also; or (ii) whenever an alternative is uniquely worst in some ordering belonging to D|A, it is not medium in any ordering belonging to D|A unless it is best also. Satisfaction of LSEVR over a triple of alternatives A requires fulfilment of extremal value restriction only over orderings of the same Latin Square and not necessarily over the set of all orderings D|A. WLSEVR is even weaker than LSEVR and requires fulfilment of LSEVR only when the set of orderings contains weak Latin Squares.
3. Salles (1976) considered a subclass of simple games (satisfying his assumptions 1 and 2). For the subclass he derives maximal sufficient conditions for quasi-transitivity. He shows that for the subclass in question each of the conditions (i) dichotomous preferences (DP), (ii) value restriction (VR) and (iii) cyclical dependence (CD) is sufficient for quasi-transitivity and that there exists a simple game for which the union of DP, VR and CD is necessary for quasi-transitivity.