Special Majority Rules: Necessary and Sufficient Condition for Quasi-Transitivity with Quasi-Transitive Individual Preferences

Satish K. Jain
Centre for Economic Studies and Planning
Jawaharlal Nehru University
New Delhi - 110067
INDIA

Abstract

A condition on preferences called strict Latin Square partial agreement is introduced and is shown to be necessary and sufficient for quasi-transitivity of the social weak preference relation generated by any special majority rule, under the assumption that individual preferences themselves are quasi-transitive.

SPECIAL MAJORITY RULES : NECESSARY AND SUFFICIENT CONDITION FOR QUASI-TRANSITIVITY WITH QUASI-TRANSITIVE INDIVIDUAL PREFERENCES

Satish K. Jain

An important problem in the context of social decision rules which in some situations fail to yield rational (transitive, quasi-transitive or acyclic) social weak preference relations is that of characterizing configurations of individual weak preference relations which always give rise to rational social weak preference relations. For several classes of social decision rules such characterizations have been obtained. Under the assumption that individual weak preference relations are transitive, necessary and sufficient conditions have been obtained for quasi-transitivity and transitivity under the simple majority rule by Inada [3] and Sen and Pattanaik [7], for transitivity under the simple non-minority rule by Fine [1], for quasi-transitivity and transitivity under the special majority rules by Jain [5] and for quasi-transitivity and transitivity under the non-minority rules by Jain [6]. For the case when individual weak preference relations are quasi-transitive, Inada [4] and Fishburn [2] have obtained necessary and sufficient conditions for quasi-transitivity under the simple majority rule. They have shown that the satisfaction of dichotomous preferences or antagonistic preferences or generalized limited agreement or generalized value-restriction over every triple of alternatives is necessary as well as sufficient for quasi-transitivity under the simple majority rule.

In this paper we consider the class of special majority rules. The term 'special' here is used to signify the fact that the majority required is greater than the simple majority. The simple majority rule declares an alternative x to be better than another alternative y if and only if the number of individuals who prefer x to y is greater than half of the total of those who prefer x to y and those who prefer y to x. Analogously we can define, corresponding to any fraction p lying strictly between $\frac{1}{2}$ and 1, a special majority rule (p - majority rule) by requiring that an alternative x be declared socially preferred to another alternative y if and only if the number of individuals who prefer x to y is greater than p of the total of those who have strict preferences between x and y. Thus, like the simple majority rule, in the case of special majority rules also, the individuals who are indifferent between two alternatives are not relevant when determining social preference between those two alternatives.[1] In [5] it is shown that, under the assumption that individual weak preference relations are orderings, the necessary and sufficient conditions for quasi-transitivity under the special majority rules are identical to the necessary and sufficient conditions for quasi-transitivity under the simple majority rule. The necessary and sufficient condition for transitivity under the special majority rules, however, is different from the condition of extremal restriction which is necessary and sufficient for transitivity under the simple majority rule. It is shown that a condition called strong value-restriction is necessary and sufficient for transitivity under all special majority rules. Strong value-restriction is a more stringent requirement than the condition of extremal restriction.

In this paper we show that, under the assumption that individual weak preference relations are quasi-transitive, a necessary and sufficient condition for quasi-transitivity under every special majority rule is that a condition called the strict Latin Square partial agreement is satisfied over every triple of alternatives. The strict Latin Square partial agreement requires that in case the set of individual weak preference relations contains a strict Latin Square involving a strong ordering or an intransitive weak preference relation then the generalized limited agreement must hold for all those individuals whose weak preference relations belong to the strict Latin Square in question. Therefore, the strict Latin Square partial agreement can be interpreted as a weakened version of the generalized limited agreement condition. The weakening is two-fold: (1) 'generalized limited agreement' is required only when the set of individual weak preference relations contains a strict

Latin Square involving a strong ordering or an intransitive weak preference relation ; (2) 'generalized limited agreement' is required only among those individuals whose weak preference relations belong to the strict Latin Square in question.

Notation and Definitions

The set of mutually exclusive alternatives will be denoted by S. The cardinality n of S will be assumed to be at least 3. We denote the set of individuals by L and the number of individuals by N. Each individual $i \in L$ will be assumed to have a reflexive, connected and quasi-transitive weak preference relation R_i defined over S. The reflexive and connected social weak preference relation generated by a special majority rule will be designated by R. The symmetric and asymmetric parts of R_i are denoted by I_i and P_i respectively and those of R by I and P respectively. N ( ) stands for the number of people holding the preferences specified in the parentheses and N_k for the number of people holding the kth preferences.

Special Majority Rules: $\forall x, y \in S : xRy$ iff $\sim[N(yP_i x) > p\,[N(xP_i y) + N(yP_i x)]]$, where p is a fraction such that $\frac{1}{2} < p < 1$. For $P = \frac{2}{3}$ we obtain the familiar two-thirds majority rule.

We define an individual to be concerned over a triple of alternatives iff he is not indifferent over every pair of alternatives belonging to the triple; otherwise he is unconcerned. For individual i, in the triple $\{x, y, z\}$, x is best iff $(xR_i y \wedge xR_i z)$, medium iff $(yR_i x \wedge xR_i z) \vee (zR_i x \wedge xR_i y)$, worst iff $(yR_i x \wedge zR_i x)$, proper best iff $(xP_i y \wedge xR_i z) \vee (xR_i y \wedge xP_i z)$, proper medium iff $(yP_i x \wedge xR_i z) \vee (yR_i x \wedge xP_i z) \vee (zP_i x \wedge xR_i y) \vee (zR_i x \wedge xP_i y)$ and proper worst iff $(yP_i x \wedge zR_i x) \vee (yR_i x \wedge zP_i x)$.

If an R_i or R is transitive it will be written in the usual way; otherwise the relation between every pair of alternatives will be written separately. There are 19 logically possible quasi-transitive R_i over a triple $\{x, y, z\}$ listed below :

1.	$xP_i yP_i z$	11.	$zP_i xI_i y$
2.	$yP_i zP_i x$	12.	$zI_i xP_i y$
3.	$zP_i xP_i y$	13.	$xI_i yI_i z$
4.	$xP_i zP_i y$	14.	$xP_i y$, $yI_i z$, $xI_i z$
5.	$zP_i yP_i x$	15.	$yP_i z$, $zI_i x$, $yI_i x$
6.	$yP_i xP_i z$	16.	$zP_i x$, $xI_i y$, $zI_i y$
7.	$xP_i yI_i z$	17.	$xP_i z$, $zI_i y$, $xI_i y$
8.	$xI_i yP_i z$	18.	$zP_i y$, $yI_i x$, $zI_i x$
9.	$yP_i zI_i x$	19.	$yP_i x$, $xI_i z$, $yI_i z$
10.	$yI_i zP_i x$		

Relations (1) to (13) are transitive and the remaining 6 are quasi-transitive with intransitive indifference. Except (13), all other R_i are concerned.

Latin Square (LS): $\{R_i, R_j, R_k\}$ form a Latin Square over a triple $\{x,y,z\}$ iff R_i, R_j, R_k are concerned over $\{x,y,z\}$ and there exist distinct $a,b,c \in \{x,y,z\}$ such that in R_i, a is best, b medium and c worst; in R_j, b best, c medium and a worst; and in R_k, c best, a medium and b worst. This Latin Square will be denoted by LS(abca).

Strict Latin Square (SLS): $\{R_i, R_j, R_k\}$ form a strict Latin Square over a triple $\{x,y,z\}$ iff there exist distinct $a,b,c \in \{x,y,z\}$ such that in R_i, a is best, b proper medium and c worst; in R_j, b best, c proper medium and a worst; and in R_k, c best, a proper medium and b worst. This strict Latin Square will be denoted by SLS (abca).

Two points should be noted about these definitions: (i) If all R_i are transitive then SLS is equivalent to LS (ii) R_i, R_j, R_k need not be distinct. Over a triple $\{x,y,z\}$ there are two logically possible strict Latin Squares, SLS (xyzx) and SLS (xzyx). The set of all logically possible quasi-transitive R_i of the SLS (xyzx) will be denoted by T(xyzx) and the set of all logically possible quasi-transitive R_i of the SLS (xzyx) by T (xzyx).

(1), (2), (3), (7), (8), (9), (10), (11), (12), (14), (15) and (16) belong to $T(xyzx)$ and (4), (5), (6), (7), (8), (9), (10), (11), (12), (17), (18) and (19) belong to $T(xzyx)$.

Now we define the condition of strict Latin Square partial agreement.

Strict Latin Square Partial Agreement (SLSPA): A set π of R_i satisfies SLSPA over $\{x,y,z\}$ iff the following holds: If there exists a strict Latin Square over $\{x,y,z\}$, say SLS (xyzx), involving a strong ordering or intransitive[2] R_i, then there exist distinct $a,b \in \{x,y,z\}$ such that $\forall R_i \in \pi \cap T(xyzx) : aR_ib$ and $\forall$ intransitive $R_i \in \pi \cap T(xyzx)$: aP_ib .

From the definition it follows that SLSPA is violated over $\{x,y,z\}$ iff the restriction to $\{x,y,z\}$ of the set π of individual weak preference relations contains one of the following 10 sets, except for a formal interchange of alternatives.

(A) 1. xP_iyP_iz
2. yP_izP_ix
3. zP_ixP_iy

(B) 1. xP_iyP_iz
2. yP_izP_ix
3. zP_ixI_iy

(C) 1. xP_iyP_iz
2. yP_izP_ix
3. zI_ixP_iy

(D) 1. xP_iyP_iz
2. yP_izI_ix
3. zP_ixI_iy

(E) 1. xP_iyP_iz
2. yI_izP_ix
3. zI_ixP_iy

(F) 1. xP_iyP_iz
2. yI_izP_ix
3. zP_ixI_iy

(G) 1. xP_iy, yI_iz, xI_iz
2. yP_izP_ix

(H) 1. xP_iy, yI_iz, xI_iz
2. yP_izI_ix

(I) 1. xP_iy, yI_iz, xI_iz
2. yI_izP_ix

(J) 1. xP_iy, yI_iz, xI_iz
2. yP_iz, zI_ix, yI_ix

Necessity and Sufficiency of SLSPA for Quasi-Transitivity

Theorem: Let p be any fraction $\in (\frac{1}{2}, 1)$ and f the special majority rule associated to p. Then, given that individual weak preference relations are quasi-transitive, a necessary[3] and sufficient condition for quasi-transitivity of the social weak preference relation generated by f is that the set of individual weak preference relations satisfies the condition of strict Latin Square partial agreement over every triple of alternatives.

Proof: Sufficiency

Suppose quasi-transitivity is violated. Then for some $x, y, z \in S$ we must have xPy, yPz and zRx.

$$xPy \longrightarrow N(xP_iy) > p\,[N(xP_iy) + N(yP_ix)]$$

$$\longrightarrow N(xP_iy) > \frac{p}{1-p}\,N(yP_ix)$$

$$\longrightarrow N_1+N_3+N_4+N_7+N_{12}+N_{14} > \frac{p}{1-p}\,[N_2+N_5+N_6+N_9+N_{10}+N_{19}] \quad (1)$$

$$yPz \longrightarrow N(yP_iz) > \frac{p}{1-p}\,N(zP_iy)$$

$$\longrightarrow N_1+N_2+N_6+N_8+N_9+N_{15} > \frac{p}{1-p}\,[N_3+N_4+N_5+N_{11}+N_{12}+N_{18}] \quad (2)$$

$$zRx \longrightarrow N(xP_iz) \leq p\,[N(xP_iz) + N(zP_ix)]$$

$$\longrightarrow N(zP_ix) \geq \frac{1-p}{p}\,N(xP_iz)$$

$$\longrightarrow N_2+N_3+N_5+N_{10}+N_{11}+N_{16} \geq \frac{1-p}{p}\,[N_1+N_4+N_6+N_7+N_8+N_{17}] \quad (3)$$

Multiplying (3) by $\frac{p}{1-p}$ we get,

$$\frac{p}{1-p}\left[N_2+N_3+N_5+N_{10}+N_{11}+N_{16}\right] \geq N_1+N_4+N_6+N_7+N_8+N_{17} \qquad (4)$$

Adding (1) and (2) we obtain,

$$2N_1+N_7+N_8+N_{14}+N_{15} > \frac{p}{1-p}\left[2N_5+N_{10}+N_{11}+N_{18}+N_{19}\right] +$$

$$\frac{2p-1}{1-p}\left[N_2+N_3+N_4+N_6+N_9+N_{12}\right] \qquad (5)$$

As $\frac{1}{2} < p < 1$,

$(5) \longrightarrow 2N_1+N_7+N_8+N_{14}+N_{15} > 0$

$\longrightarrow N_1+N_7+N_8+N_{14}+N_{15} > 0$

$\longrightarrow (\exists \text{ concerned } i : xR_i yR_i z) \vee (\exists i: xP_i y \wedge yI_i z \wedge xI_i z) \vee (\exists i: yP_i z \wedge zI_i x \wedge yI_i x)$ (6)

$\longrightarrow \exists i:$ (x best, y proper medium, z worst) (7)

Adding (2) and (4) we get,

$$\frac{1}{1-p} N_2+N_9+\frac{p}{1-p} N_{10} + N_{15} + \frac{p}{1-p} N_{16} > \frac{1}{1-p} N_4 + N_7 +$$

$$\frac{p}{1-p} N_{12} + N_{17} + \frac{p}{1-p} N_{18} \qquad (8)$$

$(8) \longrightarrow \frac{1}{1-p} N_2 + N_9 + \frac{p}{1-p} N_{10} + N_{15} + \frac{p}{1-p} N_{16} > 0$

$\longrightarrow N_2+N_9+N_{10}+N_{15}+N_{16} > 0$

$\longrightarrow$ ($\exists$ concerned i : $yR_i zR_i x$) v ($\exists$ i : $yP_i z \wedge$ $zI_i x \wedge yI_i x$) v ($\exists$ i: $zP_i x \wedge xI_i y \wedge zI_i y$) (9)

$\longrightarrow$ $\exists$ i: (y best, z proper medium, x worst) (10)

Adding (4) and (1) we obtain,

$$\frac{1}{1-p} N_3 + \frac{p}{1-p} N_{11}+N_{12}+N_{14}+ \frac{p}{1-p} N_{16} > \frac{1}{1-p} N_6+N_8+ \frac{p}{1-p} N_9 + N_{17} + \frac{p}{1-p} N_{19} \quad (11)$$

(11) $\longrightarrow$ $\frac{1}{1-p} N_3 + \frac{p}{1-p} N_{11}+N_{12}+N_{14} + \frac{p}{1-p} N_{16} > 0$

$\longrightarrow$ $N_3+N_{11}+N_{12}+N_{14}+N_{16} > 0$

$\longrightarrow$ ($\exists$ concerned i : $zR_i xR_i y$) v ($\exists$ i : $xP_i y \wedge$ $yI_i z \wedge xI_i z$) v ($\exists$ i : $zP_i x \wedge xI_i y \wedge zI_i y$) (12)

$\longrightarrow$ $\exists$ i (z best, x proper medium, y worst) (13)

(7) $\wedge$ (10) $\wedge$ (13) $\longrightarrow$ the set π of R_i contains SLS (xyzx) (14)

Adding (1), (2) and (4) we obtain,

$$N_1+N_2+N_3+N_{14}+N_{15} + \frac{p}{1-p} N_{16} > \frac{p}{1-p} (N_4+N_5+N_6) + \frac{2p-1}{1-p} (N_9+N_{12}) + N_{17} + \frac{p}{1-p} (N_{18}+N_{19}) \quad (15)$$

As $\frac{1}{2} < p < 1$,

(15) $\longrightarrow$ $N_1+N_2+N_3+N_{14}+N_{15} + \frac{p}{1-p} N_{16} > 0$

$\longrightarrow$ $N_1+N_2+N_3+N_{14}+N_{15}+N_{16} > 0$

$\longrightarrow$ $\exists$ a strong ordering $\in \pi \cap T(xyzx)$ v $\exists$ an intransitive $R_i \in \pi \cap T(xyzx)$ (16)

(6) $\longrightarrow$ $\exists R_i \in \pi \cap T(xyzx) : xP_i z$ v $\exists$ intransitive $R_i \in \pi \cap T(xyzx) : xI_i z$ (17)

(9) $\longrightarrow$ $\exists R_i \in \pi \cap T(xyzx) : yP_i x$ v $\exists$ intransitive $R_i \in \pi \cap T(xyzx) : yI_i x$ (18)

(12) $\longrightarrow$ $\exists R_i \in \pi \cap T(xyzx) : zP_i y$ v $\exists$ intransitive $R_i \in \pi \cap T(xyzx) : zI_i y$ (19)

If there exists an intransitive $R_i \in \pi \cap T(xyzx)$ then (17), (18) and (19) imply that there do not exist distinct $a,b \in \{x,y,z\}$ such that $\forall R_i \in \pi \cap T(xyzx) : aR_i b$ and $\forall$ intransitive $R_i \in \pi \cap T(xyzx) : aP_i b$ (20)

Now suppose that there does not exist an intransitive $R_i \in \pi \cap T(xyzx)$. Add (5) and (8), (8) and (11), and (11) and (5) to obtain the following 3 inequalities respectively.

$$2N_1 + 2N_2 + N_8 + \frac{2-3p}{1-p} N_9 + N_{14} + 2N_{15} + \frac{p}{1-p} N_{16} > \frac{2p-1}{1-p} N_3 + \frac{2p}{1-p} N_4 + \frac{2p}{1-p} N_5 + \frac{2p-1}{1-p} N_6 + \frac{p}{1-p} N_{11} + \frac{3p-1}{1-p} N_{12} + N_{17} + \frac{2p}{1-p} N_{18} + \frac{p}{1-p} N_{19} \qquad (21)$$

$$\frac{1}{1-p} (N_2+N_3) + \frac{p}{1-p} (N_{10} + N_{11}) + N_{14} + N_{15} + \frac{2p}{1-p} N_{16} > \frac{1}{1-p} (N_4 + N_6) + N_7 + N_8 + \frac{2p-1}{1-p} (N_9 + N_{12}) + 2N_{17} + \frac{p}{1-p} (N_{18} + N_{19}) \qquad (22)$$

$$2N_1 + 2N_3 + N_7 + \frac{2-3p}{1-p} N_{12} + 2N_{14} + N_{15} + \frac{p}{1-p} N_{16} > \frac{2p-1}{1-p} (N_2 + N_4) + \frac{2p}{1-p} (N_5 + N_6) + \frac{3p-1}{1-p} N_9 + \frac{p}{1-p} N_{10} + N_{17} + \frac{p}{1-p} N_{18} + \frac{2p}{1-p} N_{19} \qquad (23)$$

As $\frac{1}{2} < p < 1$ and $N_{14} = N_{15} = N_{16} = 0$ by supposition,

$$(21) \longrightarrow N_1 + N_2 + N_8 + N_9 > 0$$

$$\longrightarrow \exists R_i \in \pi \cap T(xyzx) : yP_i z \qquad (24)$$

$$(22) \longrightarrow N_2 + N_3 + N_{10} + N_{11} > 0$$

$$\longrightarrow \exists R_i \in \pi \cap T(xyzx) : zP_i x \quad (25)$$

$$(23) \longrightarrow N_1 + N_3 + N_7 + N_{12} > 0$$

$$\longrightarrow \exists R_i \in \pi \cap T(xyzx) : xP_i y \quad (26)$$

Under the hypothesis that there does not exist an $R_i \in \pi \cap T(xyzx)$ which is intransitive, (24), (25) and (26) together with (17), (18) and (19) imply that there do not exist distinct $a,b \in \{x,y,z\}$ such that $\forall R_i \in \pi \cap T(xyzx)$: $aR_i b$ and $\forall$ intransitive $R_i \in \pi \cap T(xyzx) : aP_i b$ (27)

(14), (16), (20) and (27) establish that SLSPA is violated. Thus violation of quasi-transitivity implies violation of SLSPA, i.e., SLSPA is sufficient for quasi-transitivity.

<u>Necessity</u>:

As noted on pages 9-10, SLSPA is violated over a triple $\{x,y,z\}$ iff the set of R_i contains one of the ten sets (A) - (J), except for a formal interchange of alternatives.

Therefore, for proving the necessity of SLSPA for quasi-transitivity it suffices to show that for each of these sets there exists an assignment of individuals which results in violation of quasi-transitivity.

For appropriate N, take for (A), (B) and (F), $N_1 = pN$ and $N_2 = N_3 = \frac{(1-p)N}{2}$; for (C), $N_1 = N_3 = \frac{(1-p)N}{2}$ and $N_2 = pN$; for (D) $M \geq \frac{p}{1-p}$, $N > \frac{M+p}{p(1-p)}$, $N_1 = pN-M$, $N_2 = M+1$ and $N_3 = (1-p)N-1$; for (E), $M \geq \frac{p}{1-p}$, $N > \frac{M+p}{p(1-p)}$, $N_1 = pN-M$, $N_2 = (1-p)N-1$ and $N_3 = M+1$; for (G) and (J), $N_1 = N_2$; and for (H) and (I), $N_1 = pN+1$ and $N_2 = (1-p)N-1$. This results, for (A), (B), (D), (E), (F), (H) and (J) in xPy, yPz and $\backsim(xPz)$; for (C) and (G) in yPz, zPx and $\backsim(yPx)$; and for (I) in zPx, xPy and $\backsim(zPy)$.

Concluding Remarks

A careful reading of the proof shows that it is valid for the case $p = \frac{1}{2}$ also. This implies that SLSPA is necessary and sufficient for quasi-transitivity under the method of majority decision also. Therefore by putting $p = \frac{1}{2}$ in the proof of the theorem one obtains an alternative proof of the Inada-Fishburn theorem. In view of Inada-Fishburn result it follows that SLSPA is logically equivalent to the union of dichotomous preferences, antagonistic preferences, generalized limited agreement and generalized value-restriction.

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