Necessary and Sufficient Conditions for Quasi-Transitivity and Transitivity of Special Majority Rules

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Abstract

It is shown that for every special majority rule (i) value-restriction, limited agreement and weakly antagonistic preferences constitute a set of necessary and sufficient conditions for quasi-transitivity of the social preference relation (ii) strong value restriction, a condition stronger than both value-restriction and extremal restriction, is necessary and sufficient for transitivity of the social preference relation.

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In this paper we establish necessary and sufficient conditions for quasi-transitivity and transitivity of social preference relation generated by any special majority rule. It is shown that for every special majority rule value-restriction, limited agreement and weakly antagonistic preferences constitute a set of necessary and sufficient conditions for quasi-transitivity of the social preference relation. Thus, conditions for quasi-transitivity of special majority rules are the same as that of the simple majority rule. For transitivity of social preference relation generated by any special majority rule, a condition introduced in this paper called strong value restriction is shown to be both necessary and sufficient. Strong value restriction is a more demanding requirement than either value-restriction or extremal restriction. Therefore, the extremal restriction which is necessary and sufficient for transitivity of the simple majority rule is necessary but not sufficient for transitivity of special majority rules.

Restrictions on Preferences

The set of social alternatives would be denoted by S. The cardinality n of S would be assumed to be finite and greater than 2. The set of individuals and the number of individuals are designated by L and N respectively. N ( ) would stand for the number of individuals holding the preferences specified in the parentheses, and N_k for the number of individuals holding the k-th preference ordering. Each individual $i \in L$ is assumed to have an ordering R_i defined over S. The symmetric and asymmetric parts of R_i are denoted by I_i and P_i respectively. The social preference relation is denoted by R and its symmetric and asymmetric components by I and P respectively.

Special Majority Rules : $\forall\, x, y \in S$: $x \, R \, y$ iff $N(y \, P_i \, x) \leq p \, [N(x \, P_i \, y) + N(y \, P_i \, x)]$, where p is a fraction such that $\frac{1}{2} < p < 1$. For $p = \frac{2}{3}$ we obtain the familiar two-thirds majority rule.

An individual is defined to be concerned with respect to a triple iff he is not indifferent over every pair of alternatives belonging to the triple; otherwise he is unconcerned. For individual i, in the triple $\{x, y, z\}$, x is best iff $(x R_i y \wedge x R_i z)$; medium iff $(y R_i x R_i z \vee z R_i x R_i y)$; worst iff $(y R_i x \wedge z R_i x)$; uniquely best iff $(x P_i y \wedge x P_i z)$; uniquely medium iff $(y P_i x P_i z \vee z P_i x P_i y)$; and uniquely worst iff $(y P_i x \wedge z P_i x)$.

Now we define several restrictions which specify the permissible sets of individual orderings. All these restrictions are defined over triples of alternatives.

Value-Restriction (VR) : It holds over a triple iff there is an alternative in the triple such that all concerned individuals agree that it is not best or it is not medium or it is not worst.

Limited Agreement (LA) : It holds over $\{x, y, z\}$ iff there exist distinct $a, b \in \{x, y, z\}$ such that $\forall i \in L$: $a R_i b$.

Dichotomous Preferences (DP) : It holds over a triple iff no individual has a strong ordering over the triple.

Weakly Antagonistic Preferences (WAP)[1]: $\forall a, b, c \in \{x,y,z\}$: $[(\exists i : a P_i b P_i c) \longrightarrow \forall i : (a P_i b P_i c \vee c P_i b P_i a \vee a I_i c)]$.

Strong Value Restriction (SVR) : It is satisfied over a triple iff there exists (i) an alternative such that it is best in every R_i or (ii) an alternative such that it is worst in every R_i or (iii) an alternative such that it is uniquely medium in every concerned R_i or (iv) a pair of distinct alternatives such that every individual is indifferent between the alternatives of the pair.

More formally, SVR holds over $\{x, y, z\}$ iff there exist distinct a, b, c $\in \{x, y, z\}$ such that $[\ \forall i :$ $(a\ R_i\ b \wedge a\ R_i\ c) \vee \forall i: (b\ R_i\ a \wedge c\ R_i\ a) \vee \forall$ concerned $i:$ $(b\ P_i\ a\ P_i\ c \vee c\ P_i\ a\ P_i\ b) \vee \forall i: a\ I_i\ b\ ]$.

Conditions for Quasi-Transitivity

Lemma 1 : For every special majority rule, a sufficient condition for quasi-transitivity of the social preference relation is that DP holds over every triple of alternatives.

Proof : Satisfaction of DP over a triple $\{x, y, z\}$ implies that the set of permissible orderings must be a subset of the following 7 orderings,

1. $x \; P_i \; y \; I_i \; z$

2. $y \; I_i \; z \; P_i \; x$

3. $y \; P_i \; x \; I_i \; z$

4. $x \; I_i \; z \; P_i \; y$

5. $z \; P_i \; x \; I_i \; y$

6. $x \; I_i \; y \; P_i \; z$

7. $x \; I_i \; y \; I_i \; z$

Because of symmetry it is sufficient to show that $x \; P \; y$ and $y \; P \; z$ imply $x \; P \; z$.

$$x \; P \; y \longrightarrow N_1 + N_4 > p\,(N_1 + N_2 + N_3 + N_4)$$

$$y \; P \; z \longrightarrow N_3 + N_6 > p\,(N_3 + N_4 + N_5 + N_6)$$

Combining the two inequalities we obtain,

$$N_1 + N_3 + N_4 + N_6 > p\,(N_1 + N_2 + N_5 + N_6) + 2p\,(N_3 + N_4)$$

$$\longrightarrow N_1 + N_6 > p\,(N_1 + N_2 + N_5 + N_6) + (2p-1)\,(N_3 + N_4)$$

$$\longrightarrow N_1 + N_6 > p\,(N_1 + N_2 + N_5 + N_6), \text{ as } p > \frac{1}{2}$$

$$\longrightarrow x \; P \; z \,.$$

Theorem 1 : For every special majority rule, a sufficient condition for quasi-transitivity of the social preference relation is that WAP is satisfied over every triple of alternatives.

Proof: If no individual has a strong ordering over $\{x, y, z\}$ then quasi-transitivity follows from lemma 1. For non-trivial fulfilment of WAP assume, without any loss of generality, that someone has the ordering $x \, P_i \, y \, P_i \, z$. Then it follows that the set of permissible orderings must be a subset of the following 5 orderings,

1. $x \, P_i \, y \, P_i \, z$
2. $z \, P_i \, y \, P_i \, x$
3. $y \, P_i \, x \, I_i \, z$
4. $x \, I_i \, z \, P_i \, y$
5. $x \, I_i \, y \, I_i \, z$

Quasi-transitivity is violated iff exactly one of the following two cycles holds with at least 2 of the R being P,

$x \, R \, y \wedge y \, R \, z \wedge z \, R \, x$ (Forward cycle)

$y \, R \, x \wedge x \, R \, z \wedge z \, R \, y$ (Backward cycle).

Suppose the forward cycle holds with at least 2 of the R being P. First suppose that z P x obtains

$$z\ P\ x \longrightarrow N_2 > p\ (N_1 + N_2)$$

$$\longrightarrow N_2 > \frac{p}{1 - p}\ N_1$$

$$\longrightarrow N_2 > N_1\ , \text{ as } p > \frac{1}{2}\ .$$

Now,

$$(x\ R\ y \wedge y\ R\ z) \longrightarrow N_2 + N_3 \leqslant p\ (N_1 + N_2 + N_3 + N_4)$$

$$\text{and } N_2 + N_4 \leqslant p\ (N_1 + N_2 + N_3 + N_4)$$

$$\longrightarrow N_1 + N_3 \leqslant p\ (N_1 + N_2 + N_3 + N_4)$$

$$\text{and } N_1 + N_4 \leqslant p\ (N_1 + N_2 + N_3 + N_4),$$

$$\text{as } N_2 > N_1$$

$$\longrightarrow z\ R\ y \wedge y\ R\ x$$

$$\longrightarrow x\ I\ y \wedge y\ I\ z\ .$$

Therefore, if z P x holds then it is impossible for the forward cycle to hold with at least 2 of R being P. The only remaining possibility is $x\ P\ y \wedge y\ P\ z \wedge x\ I\ z$. However,

$$x\ P\ y \wedge y\ P\ z \longrightarrow N_1 + N_4 > p\ (N_1 + N_2 + N_3 + N_4)$$

$$\text{and } N_1 + N_3 > p\ (N_1 + N_2 + N_3 + N_4)$$

$$\longrightarrow 2 N_1 > 2 p (N_1 + N_2) + (2 p - 1) (N_3 + N_4)$$

$$\longrightarrow N_1 > p (N_1 + N_2), \text{ as } p > \frac{1}{2}$$

$$\longrightarrow x P z ,$$

which contradicts $x I z$. Therefore it is impossible for the forward cycle to hold with at least 2 of R being P. Analogously it can be shown that the backward cycle cannot hold with at least 2 of R being P. So R must be quasi-transitive.

Lemma 2 : A set of orderings violates all three restrictions VR, LA and WAP iff it includes one of the following six 3-ordering sets, except for a formal interchange of alternatives,[2]

(A)
$x P_i y P_i z$
$y P_i z P_i x$
$z P_i x P_i y$

(B)
$x P_i y P_i z$
$y P_i z P_i x$
$z P_i x I_i y$

(C)
$x P_i y P_i z$
$y P_i z P_i x$
$z I_i x P_i y$

(D)
$x P_i y P_i z$
$y P_i z I_i x$
$z P_i x I_i y$

(E) $x \, P_i \, y \, P_i \, z$
$y \, I_i \, z \, P_i \, x$
$z \, P_i \, x \, I_i \, y$

(F) $x \, P_i \, y \, P_i \, z$
$y \, I_i \, z \, P_i \, x$
$z \, I_i \, x \, P_i \, y$

Proof : It is well known that a set of orderings violates VR iff it contains a set of 3 concerned orderings forming a Latin Square,[3]

Latin Square I

$x \, R_i \, y \, R_i \, z$
$y \, R_i \, z \, R_i \, x$
$z \, R_i \, x \, R_i \, y$

Latin Square II

$x \, R_i \, z \, R_i \, y$
$z \, R_i \, y \, R_i \, x$
$y \, R_i \, x \, R_i \, z$

There are in all 54 such 3-ordering sets. However, it is sufficient to consider the following 11 sets as the remaining ones can be obtained from these by a formal interchange of alternatives,

(1) $x \, P_i \, y \, P_i \, z$
$y \, P_i \, z \, P_i \, x$
$z \, P_i \, x \, P_i \, y$

(2) $x \, P_i \, y \, P_i \, z$
$y \, P_i \, z \, P_i \, x$
$z \, P_i \, x \, I_i \, y$

(3) $x \, P_i \, y \, P_i \, z$
$y \, P_i \, z \, P_i \, x$
$z \, I_i \, x \, P_i \, y$

(4) $x \, P_i \, y \, P_i \, z$
$y \, P_i \, z \, I_i \, x$
$z \, P_i \, x \, I_i \, y$

(5) $x\ P_i\ y\ P_i\ z$
$y\ P_i\ z\ I_i\ x$
$z\ I_i\ x\ P_i\ y$

(6) $x\ P_i\ y\ P_i\ z$
$y\ I_i\ z\ P_i\ x$
$z\ P_i\ x\ I_i\ y$

(7) $x\ P_i\ y\ P_i\ z$
$y\ I_i\ z\ P_i\ x$
$z\ I_i\ x\ P_i\ y$

(8) $x\ P_i\ y\ I_i\ z$
$y\ P_i\ z\ I_i\ x$
$z\ P_i\ x\ I_i\ y$

(9) $x\ P_i\ y\ I_i\ z$
$y\ P_i\ z\ I_i\ x$
$z\ I_i\ x\ P_i\ y$

(10) $x\ P_i\ y\ I_i\ z$
$y\ I_i\ z\ P_i\ x$
$z\ I_i\ x\ P_i\ y$

(11) $x\ I_i\ y\ P_i\ z$
$y\ I_i\ z\ P_i\ x$
$z\ I_i\ x\ P_i\ y$

(1), (2), (3), (4), (6) and (7) are the same as A, B, C, D, E and F respectively. Consider (5). Both LA and WAP are satisfied. To violate LA one has to include ($y\ P_i\ z\ P_i\ x \ \vee\ y\ I_i\ z\ P_i\ x \ \vee\ z\ P_i\ y\ P_i\ x \ \vee\ z\ P_i\ y\ I_i\ x \ \vee\ z\ P_i\ x\ P_i\ y$). Inclusion of any of these orderings excepting that of $z\ P_i\ y\ P_i\ x$ would imply a violation of WAP also and in each case one of the six sets would be contained in the set of R_i. If we include $z\ P_i\ y\ P_i\ x$ then WAP is violated iff a concerned ordering not already contained in the set is included. If a strong ordering is included then the set contains B or C. If a weak ordering is included then D or E or F is contained. Now consider (8) which satisfies WAP but violates LA. To violate WAP a strong ordering must be included. Because of symmetry it suffices to consider the case when $x\ P_i\ y\ P_i\ z$ is included. With the inclusion of $x\ P_i\ y\ P_i\ z$ the set contains D. The case of (11) is similar. Next we consider (9). Both WAP and LA are satisfied. To violate LA we have to include $(y\ P_i\ z\ P_i\ x\ \vee\ y\ I_i\ z\ P_i\ x\ \vee\ z\ P_i\ y\ P_i\ x\ \vee\ z\ P_i\ y\ I_i\ x\ \vee\ z\ P_i\ x\ P_i\ y)$. If $y\ P_i\ z\ P_i\ x$ or $z\ P_i\ y\ P_i\ x$ or $z\ P_i\ x\ P_i\ y$ is included then WAP is also violated and the set includes D or E or F. If $y\ I_i\ z\ P_i\ x$ or $z\ P_i\ y\ I_i\ x$ is included then WAP continues to be satisfied. WAP would be violated iff a strong ordering is included. Inclusion of a strong ordering makes the set contain D or E or F. Demonstration for the case (11) is analogous. Proof is completed by noting that all the six sets violate all three restrictions.

Theorem 2 : For every special majority rule, a necessary and sufficient condition for quasi-transitivity of the social preference relation is that (VR v LA v WAP) is satisfied over every triple of alternatives.

Proof : Sen [7] has shown that for the class of binary social decision rules satisfying neutrality, monotonicity and the strict Pareto-criterion, both VR and LA are sufficient conditions for quasi-transitivity of the social R. As all special majority rules are binary social decision rules satisfying monotonicity, neutrality and the strict Pareto-criterion, the sufficiency of VR and LA follows as a corollary of Sen's theorems. Sufficiency of WAP has been shown in theorem 1. In what follows we show that if a set of orderings violates all three restrictions then there exists an assignment of individuals such that R violates quasi-transitivity, establishing the necessity part. If a set of orderings violates all the three restrictions then by lemma 2 it must include one of the six sets (A) - (F) mentioned in the statement of the lemma. Therefore, it suffices to show that for each of the six sets there exists an assignment such that R violates quasi-transitivity.

For (A) take $N_1 = pN$, $N_2 = N_3 = \frac{(1-p)\ N}{2}$, for (B) $N \geqslant \frac{1}{p(1-p)}$, $N_1 = p^2N + 1$, $N_2 = p\ (1-p)\ N$, $N_3 = (1-p)N-1$, for (C) $N \geqslant \frac{1}{p(1-p)}$, $N_1 = p\ (1-p)\ N$, $N_2 = p^2N + 1$, $N_3 = (1-p)N-1$, for (D) $M \geqslant \frac{p}{1-p}$, $N > \frac{M+p}{p(1-p)}$, $N_1 = pN - M$, $N_2 = M + 1$, $N_3 = (1-p)N-1$, for (E) $M \geqslant \frac{p}{1-p}$, $N > \frac{Mp+1}{(1-p)2}$, $N_1 = (1-p)\ N-1$, $N_2 = M + 1$, $N_3 = pN - M$ and for (F) $M \geqslant \frac{p}{1-p}$, $N > \frac{M+p}{p(1-p)}$, $N_1 = pN - M$, $N_2 = (1-p)\ N-1$, $N_3 = M + 1$. This results, for (A), (B), (D) and (F) in $x\ P\ y \wedge y\ P\ z \wedge \sim(x\ p\ z)$, for (C) in $y\ P\ z \wedge z\ P\ x \wedge \sim(y\ p\ x)$ and for (E) in $z\ P\ x \wedge x\ P\ y \wedge \sim(z\ P\ y)$.

Conditions for Transitivity

Theorem 3 : For every special majority rule, a necessary and sufficient condition for transitivity of the social preference relation is that the strong value restriction holds over every triple of alternatives.

Proof :

<u>Sufficiency:</u>

Suppose transitivity is violated. Then there are x, y, z such that $x\ R\ y \wedge y\ R\ z \wedge z\ P\ x$. Let N_c denote the number of individuals who are concerned with respect to the triple $\{x, y, z\}$.

$$x \; R \; y \longrightarrow N \,(y \; P_i \; x) \leqslant p \,[N \,(x \; P_i \; y) + N(y \; P_i \; x)]$$

$$\longrightarrow N \,(x \; P_i \; y) \geqslant (1-p) \,[N \,(x \; P_i \; y) + N(y \; P_i \; x)]$$

$$\longrightarrow N \,(x \; P_i \; y) + N \,(\text{concerned } i : x \; I_i \; y) \geqslant$$

$$(1-p) \; N_c + pN \,(\text{concerned } i : x \; I_i \; y)$$

$$\longrightarrow N \,(\text{concerned } i : x \; R_i \; y) \geqslant (1-p) \; N_c \qquad (1)$$

Similarly, $y \; R \; z \longrightarrow N \,(\text{concerned } i : y \; R_i \; z) \geqslant (1-p)N_c$ (2)

$$z \; P \; x \longrightarrow N \,(z \; P_i \; x) > p \,[N \,(x \; P_i \; z) + N \,(z \; P_i \; x)]$$

$$\longrightarrow N \,(\text{concerned } i : z \; R_i \; x) > pN_c \qquad (3)$$

(1) and (3) $\longrightarrow \exists \text{concerned } i : z \; R_i \; x \; R_i \; y$ (4)

(2) and (3) $\longrightarrow \exists \text{concerned } i : y \; R_i \; z \; R_i \; x$ (5)

(4) $\longrightarrow \exists i : z \; P_i \; y$ (6)

$y \; R \; z \wedge$ (6) $\longrightarrow \exists i : y \; P_i \; z$ (7)

(5) $\longrightarrow \exists i : y \; P_i \; x$ (8)

$x \; R \; y \wedge$ (8) $\longrightarrow \exists i : x \; P_i \; y$ (9)

$z \; P \; x \longrightarrow \exists i : z \; P_i \; x$ (10)

(4) through (10) imply that SVR is violated. Thus violation of transitivity implies violation of SVR, i.e., SVR is a sufficient condition for transitivity.

<u>Necessity</u>:

It can be easily checked that SVR is violated over a triple $\{x, y, z\}$ iff the set of R_i contains one of the following 10 sets of orderings, except for a formal interchange of alternatives,

(A) $x \; P_i \; y \; P_i \; z$
$y \; P_i \; z \; P_i \; x$

(B) $x \; P_i \; y \; P_i \; z$
$z \; P_i \; x \; I_i \; y$

(C) $x \; P_i \; y \; P_i \; z$
$y \; I_i \; z \; P_i \; x$

(D) $x \; P_i \; y \; P_i \; z$
$z \; P_i \; y \; P_i \; x$
$y \; P_i \; x \; I_i \; z$

(E) $x \; P_i \; y \; P_i \; z$
$z \; P_i \; y \; P_i \; x$
$x \; I_i \; z \; P_i \; y$

(F) $x \; P_i \; y \; P_i \; z$
$y \; P_i \; x \; I_i \; z$
$x \; I_i \; z \; P_i \; y$

(G) $x \; P_i \; y \; I_i \; z$
$y \; P_i \; x \; I_i \; z$
$x \; I_i \; z \; P_i \; y$

(H) $x \; I_i \; y \; P_i \; z$
$y \; P_i \; x \; I_i \; z$
$x \; I_i \; z \; P_i \; y$

(I) $x \; P_i \; y \; I_i \; z$
$y \; P_i \; z \; I_i \; x$
$z \; P_i \; x \; I_i \; y$

(J) $x \; I_i \; y \; P_i \; z$
$y \; I_i \; z \; P_i \; x$
$z \; I_i \; x \; P_i \; y$

Therefore, for proving the necessity of SVR it suffices to show that for each of these sets there exists an assignment of individuals which results in intransitive social preference relation.

Take for (A), (B) and (C), $N_1 = N_2 = \frac{N}{2}$, for (D) and (E), $M \geqslant \frac{p}{1-p}$, $N \geqslant \frac{M}{2p-1}$, $N_1 = pN-M$, $N_2 = (1-p)N-1$, $N_3 = M + 1$, for (F), (G) and (H), $N_1 = (2p-1)N$, $N_2 = N_3 = (1-p)N$, and for (I) and (J), $N \geqslant \frac{1+p}{p(2p-1)}$, $N_1 = \frac{p}{1+p}N + 1$, $N_2 = \frac{1-p}{1+p} N$, $N_3 = \frac{p}{1+p} N-1$. This results, for (A), (C) and (D) in $x\, I\, y \wedge y\, P\, z \wedge x\, I\, z$, for (B), (E) and (I) in $x\, P\, y \wedge y\, I\, z \wedge x\, I\, z$, and for (F), (G), (H) and (J) in $x\, I\, y \wedge y\, I\, z \wedge x\, P\, z$.

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